A solution to the "exercise" in the slide p.17

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This document is a supplementary material for the first lecture (Apr 6), available from

`http://www.stat.t.u-tokyo.ac.jp/~sei/lec.html`

We first solve the problem for $N = 2$ case. This case is most important to understand.

1 The problem for $N = 2$ case

Let T_1 and T_2 be independent random variables with

$$P(T_1 > t) = P(T_2 > t) = e^{-\alpha t}, \quad t \geq 0,$$

for some $\alpha > 0$. Define U_1 and U_2 by

$$U_1 = T_{(1)} \quad \text{and} \quad U_2 = T_{(2)} - T_{(1)},$$

where[*1]

$$T_{(1)} = \min(T_1, T_2) \quad \text{and} \quad T_{(2)} = \max(T_1, T_2). \tag{1}$$

The problem is to show that U_1 and U_2 are independent, and

$$P(U_1 > t) = e^{-2\alpha t}, \quad P(U_2 > t) = e^{-\alpha t}.$$

2 Preliminaries

See the slides p.26 to p.29 (or any book on probability) for relevant definitions.

The distribution function $F(t)$ of T_i $(i = 1, 2)$ is

$$\begin{aligned} F(t) &= P(T_i \leq t) \\ &= 1 - P(T_i > t) \\ &= 1 - e^{-\alpha t} \end{aligned}$$

[*1] The random variable $\min(T_1, T_2)$ should be interpreted as the function $\omega \mapsto \min(T_1(\omega), T_2(\omega))$. Refer to the slide p.26 for the definition of random variables. The same applies to $\max(T_1, T_2)$.

for $t \geq 0$. This distribution is called *the exponential distribution.* Its density function is

$$\begin{aligned} f(t) &= \frac{dF(t)}{dt} \\ &= \alpha e^{-\alpha t}. \end{aligned}$$

The joint density function of T_1 and T_2 is, by independence,

$$\begin{aligned} f(t_1, t_2) &= f(t_1) f(t_2) \\ &= \alpha e^{-\alpha t_1} \alpha e^{-\alpha t_2}. \end{aligned}$$

We use the following two facts as mentioned in the lecture.

Lemma 1. Let T_1 and T_2 be independent and identically distributed random variables with a density function $f(t)$. Define $T_{(1)}$ and $T_{(2)}$ by Equation (1). Then the joint density function of $T_{(1)}$ and $T_{(2)}$ is

$$f_{T_{(1)}, T_{(2)}}(t_1, t_2) = \begin{cases} 2f(t_1)f(t_2) & \text{if } t_1 < t_2, \\ 0 & \text{if } t_1 > t_2. \end{cases}$$

Proof. Fix $t_1 < t_2$. Consider two intervals $J = [t_1 - \delta, t_1 + \delta]$ and $K = [t_2 - \delta, t_2 + \delta]$ for small $\delta > 0$ such that $J \cap K = \emptyset$. Then

$$\begin{aligned} P(T_{(1)} \in J,\ T_{(2)} \in K) &= P(T_1 \in J,\ T_2 \in K) + P(T_1 \in K,\ T_2 \in J) \\ &= \int_J \int_K f(t_1, t_2) dt_1 dt_2 + \int_K \int_J f(t_1, t_2) dt_1 dt_2 \\ &= \int_J \int_K f(t_1) f(t_2) dt_1 dt_2 + \int_K \int_J f(t_1) f(t_2) dt_1 dt_2 \\ &= \int_J \int_K 2 f(t_1) f(t_2) dt_1 dt_2. \end{aligned}$$

Since δ is arbitrary, we deduce that the joint density of $(T_{(1)}, T_{(2)})$ at (t_1, t_2) is $2f(t_1)f(t_2)$ whenever $t_1 < t_2$. The joint density is zero if $t_1 > t_2$ since $T_{(1)} \leq T_{(2)}$ by definition. $\square$

Lemma 2 (Change of variables formula)**.** Let (X_1, X_2) be a random vector with the joint density function $f(x_1, x_2)$, and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a one-to-one map. Define a random vector (Y_1, Y_2) by

$$(X_1, X_2) = \psi(Y_1, Y_2),$$

or equivalently $(Y_1, Y_2) = \psi^{-1}(X_1, X_2)$. Then the density function $g(y_1, y_2)$ of (Y_1, Y_2) is

$$g(y_1, y_2) = f(\psi(y_1, y_2)) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right|,$$

where $|\cdot|$ denotes the determinant and

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{pmatrix} \partial x_1 / \partial y_1 & \partial x_1 / \partial y_2 \\ \partial x_2 / \partial y_1 & \partial x_2 / \partial y_2 \end{pmatrix}.$$

Proof. For any subset $A \subset \mathbb{R}^2$, we have

$$\begin{aligned} P((Y_1, Y_2) \in A) &= P((X_1, X_2) \in \psi(A)) \\ &= \int_{\psi(A)} f(x_1, x_2) dx_1 dx_2 \\ &= \int_A f(\psi(y_1, y_2)) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| dy_1 dy_2 \end{aligned}$$

by the change of variables formula for integration. This implies the result. $\square$

3 The solution for $N = 2$ case

By Lemma 1, the joint density function of $(T_{(1)}, T_{(2)})$ is

$$\begin{aligned} f_{T_{(1)}, T_{(2)}}(t_1, t_2) &= 2f(t_1)f(t_2) \\ &= 2\alpha^2 e^{-\alpha t_1} e^{-\alpha t_2} \end{aligned}$$

if $t_1 < t_2$, and 0 otherwise.

Next we derive the joint density function $g(u_1, u_2)$ of $U_1 = T_{(1)}$ and $U_2 = T_{(2)} - T_{(1)}$ by using Lemma 2. As we can write

$$T_{(1)} = U_1, \quad T_{(2)} = U_1 + U_2,$$

we define the map ψ by

$$(t_1, t_2) = \psi(u_1, u_2) = (u_1, u_1 + u_2).$$

The Jacobian is

$$\left| \frac{\partial(t_1, t_2)}{\partial(u_1, u_2)} \right| = \left| \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right| = 1.$$

Thus we obtain

$$\begin{aligned} g(u_1, u_2) &= f_{T_{(1)}, T_{(2)}}(u_1, u_1 + u_2) \\ &= 2\alpha^2 e^{-\alpha u_1} e^{-\alpha(u_1 + u_2)} \\ &= (2\alpha e^{-2\alpha u_1})(\alpha e^{-\alpha u_2}) \end{aligned}$$

if $u_1, u_2 > 0$, and $g(u_1, u_2) = 0$ otherwise. This implies that U_1 and U_2 are independent, and their marginal density functions are

$$g_1(u_1) = 2\alpha e^{-2\alpha u_1} \quad \text{and} \quad g_2(u_2) = \alpha e^{-\alpha u_1},$$

respectively, since $\int_0^\infty g_i(u_i) du_i = 1$ for $i = 1, 2$. Finally, we have

$$P(U_1 > t) = \int_t^\infty 2\alpha e^{-2\alpha u_1} du_1 = e^{-2\alpha t}, \quad P(U_2 > t) = \int_t^\infty \alpha e^{-\alpha u_2} du_2 = e^{-\alpha t}.$$

$\square$

4 Remark

The distribution of U_1 is easy to obtain:

$$\begin{aligned}
P(U_1 > t) &= P(T_{(1)} > t) \\
&= P(\min(T_1, T_2) > t) \\
&= P(T_1 > t,\ T_2 > t) \\
&= P(T_1 > t)P(T_2 > t) \quad \text{(by independence)} \\
&= e^{-\alpha t} e^{-\alpha t} \\
&= e^{-2\alpha t}.
\end{aligned}$$

This method was not mentioned in the lecture.

We obtain the conditional distribution of U_2 given U_1 in a similar way. However, it needs a careful treatment on conditional probability and is omitted here.

The rest is about the solution for general N. It is not necessary to understand completely. Don't be afraid!

5 The problem (for general N)

Let $T_1, \ldots, T_N$ be independent random variables with

$$P(T_n > t) = e^{-\alpha t}, \quad t \geq 0, \tag{2}$$

for some $\alpha > 0$. Define $U_1, \ldots, U_N$ by

$$U_1 = T_{(1)}, \quad U_n = T_{(n)} - T_{(n-1)}, \quad n \geq 2,$$

where $T_{(n)}$ denotes the n-th smallest value in $T_1, \ldots, T_N$. The random variables $T_{(1)}, \ldots, T_{(N)}$ are sometimes called *the order statistics* of $T_1, \ldots, T_N$.

The problem is to show that $U_1, \ldots, U_N$ are independent and that

$$P(U_n > t) = e^{-(N-n+1)\alpha t}$$

for $n = 1, \ldots, N$.

6 Solution (for general N)

The joint density function $f(t_1, \dots, t_N)$ of $T_1, \dots, T_N$ is, by independence,

$$\begin{aligned} f(t_1, \dots, t_N) &= \prod_{n=1}^{N} f(t_n) \\ &= \prod_{n=1}^{N} \alpha e^{-\alpha t_n}. \end{aligned}$$

We use the following two lemmas as in the $N = 2$ case. The proof is similar and omitted.

Lemma 3. Let $T_1, \dots, T_N$ be independent and identically distributed random variables with a density function $f(t)$. Then the joint density function of the order statistics $T_{(1)}, \dots, T_{(N)}$ is given by[*2]

$$f_{T_{(1)},\dots,T_{(N)}}(t_1, \dots, t_N) = \begin{cases} N! \prod_{n=1}^{N} f(t_n) & \text{if } t_1 < t_2 < \cdots < t_N, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4. Let $\boldsymbol{X} = (X_1, \dots, X_N)$ be a random vector with the joint density function $f(\boldsymbol{x})$, and $\psi : \mathbb{R}^N \to \mathbb{R}^N$ be a one-to-one map. Define a random vector $\boldsymbol{Y} = (Y_1, \dots, Y_N)$ by

$$\boldsymbol{X} = \psi(\boldsymbol{Y}).$$

Then the joint density function $g(\boldsymbol{y})$ of $\boldsymbol{Y}$ is

$$g(\boldsymbol{y}) = f(\psi(\boldsymbol{y})) \left| \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}} \right|.$$

By Lemma 3, the joint density of $T_{(n)}$'s in our problem is

$$\begin{aligned} f_{T_{(1)},\dots,T_{(N)}}(t_1, \dots, t_N) &= N! \prod_{n=1}^{N} f(t_n) \\ &= N! \prod_{n=1}^{N} \alpha e^{-\alpha t_n} \end{aligned}$$

if $t_1 < \cdots < t_N$, and 0 otherwise.

Next we derive the joint density function $g(u_1, \dots, u_N)$ of U_n's by using Lemma 4. As we can write

$$T_{(n)} = \sum_{k=1}^{n} U_k,$$

[*2] The factorial $N!$ comes from the number of permutations of N distinct real numbers.

we define the map $\psi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\psi(u_1, \dots, u_N) = (u_1, u_1 + u_2, \dots, u_1 + \cdots + u_N).$$

The Jacobian determinant is shown to be 1. Therefore

$$\begin{aligned} g(u_1, \dots, u_N) &= f_{T_{(1)},\dots,T_{(n)}}(u_1, u_1 + u_2, \dots, u_1 + \cdots + u_N) \\ &= N! \prod_{k=1}^{N} \alpha e^{-\alpha \sum_{i=1}^{k} u_i} \\ &= N! \prod_{n=1}^{N} \alpha e^{-(N+1-n)\alpha u_n} \\ &= \prod_{n=1}^{N} (N+1-n)\alpha e^{-(N+1-n)\alpha u_n} \end{aligned} \tag{3}$$

if $u_1, \dots, u_N > 0$, and $g(u_1, \dots, u_N) = 0$ otherwise. The reason why the factorial $N!$ is shared as Equation (3) is to make each factor a density function. Equation (3) shows that $U_1, \dots, U_N$ are independent, and the density function of U_n for each n is

$$(N+1-n)\alpha e^{-\alpha(N+1-n)u_n}.$$

Finally, we have

$$\begin{aligned} P(U_n > t) &= \int_t^{\infty} (N+1-n)\alpha e^{-(N+1-n)\alpha u_n} du_n \\ &= e^{-(N+1-n)\alpha t} \end{aligned}$$

for each n. $\square$