Hints for solving recommended problems

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In this note, we first show that the Monte Carlo method is a quite powerful tool for checking your answer numerically. We deal with two problems for that purpose. Then we provide hints (or answers) for difficult problems.

Denote, for example, Problem 22 of Section 1.8 of PRP[*1] by Problem 1.8.22.

1 Use of the Monte Carlo method

Problem 1.8.22

■The problem A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from the remaining fifteen.

(a) What is the probability that this cherry contains a stone?
(b) Given that this cherry contains a stone, what is the probability that the pig consumed at least one stone.

■Answer The following two procedures cause the same probability distribution:

(i) The pig eats five cherries randomly, and then a cherry is picked randomly from the remaining 15 cherries.
(ii) A cherry is picked randomly, and then pig eats five cherries randomly from the remaining 19 cherries.

If you are skeptic about equivalence of (i) and (ii), please refer to Remark 1 below.

[*1] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, 3rd ed., Oxford University Press, 2001.

(a) According to the procedure (ii), the probability we want is obviously

$$\frac{5}{20} = 0.25.$$

(b) According to the procedure (ii), the probability that the pig consumed no stone given the condition is

$$\frac{\binom{4}{0}\binom{15}{5}}{\binom{19}{5}} = \frac{1001}{3876}.$$

Therefore the probability that the pig consumed at least one stone is

$$1 - \frac{1001}{3876} = \frac{2875}{3876} = 0.7417\cdots.$$

■**Checking by Monte Carlo** The Monte Carlo method computes an approximate value of $E[X]$ for a given random variable X. The algorithm is described as follows:

- Generate random numbers $X_1, \ldots, X_N$ having the same distribution as X.
- Then compute $(1/N)\sum_{i=1}^{N} X_i$.

Very easy!

In our problem, the targets is the probability

$$P(Y = 1) = E[I_{\{Y=1\}}]$$

for (a), and the conditional probability

$$P(X > 0 \mid Y = 1) = \frac{P(X > 0,\ Y = 1)}{P(Y = 1)} = \frac{E[I_{\{X>0,Y=1\}}]}{E[I_{\{Y=1\}}]}$$

for (b), where X and Y are the number of cherries eaten by pig and picked, respectively.

Here is an R code for checking our answer to the problem. If you are not familiar with R language, just read the comments after the symbol "#".

R code for checking Problem 1-8-22

```
N = 1e4  # number of experiments
Xs = numeric(N)  # vector of length N
Ys = numeric(N)
for(i in 1:N){
  P = sample(20, 5)   # "the pig randomly eats five cherries"
  Xs[i] = sum(P <= 5)   # the number of eaten cherries with stones.
  Q = sample((1:20)[-P], 1)   # "a cherry is picked randomly"
  Ys[i] = sum(Q <= 5)   # 1 if this cherry contains a stone, and 0 otherwise
}
A = mean(Ys)  # answer to (a)
B = mean((Xs > 0) & Ys) / mean(Ys)  # answer to (b)
```

The result was (a) 0.2548 and (b) 0.7296, which depend on the random seed. The values are close to our answer (a) 0.25 and (b) 0.7417. If you take larger N, you will get closer values.

It is also available to estimate the error. See Remark 2 below.

■Remark 1: equivalence of the procedures (i) and (ii) In general, suppose that there are N cherries in the bowl, exactly n of which have stones. The pig eats p cherries and then q cherries are picked randomly from the remaining. Let X be the number of eaten ones with stones, and Y be the number of picked ones with stones. Then we have

$$P(X=x)=\frac{\binom{n}{x}\binom{N-n}{p-x}}{\binom{N}{p}},\quad P(Y=y\mid X=x)=\frac{\binom{n-x}{y}\binom{(N-n)-(p-x)}{q-y}}{\binom{N-p}{q}}.$$

They are the hypergeometric distributions. Now their joint distribution is

$$\begin{aligned}P(X=x,\ Y=y)&=P(X=x)P(Y=y\mid X=x)\\&=\frac{\binom{n}{x}\binom{N-n}{p-x}}{\binom{N}{p}}\cdot\frac{\binom{n-x}{y}\binom{(N-n)-(p-x)}{q-y}}{\binom{N-p}{q}}\end{aligned}$$

In order to show that the procedures (i) and (ii) are equivalent, it is enough to see that $P(X=x,\ Y=y)$ is symmetric with respect to exchange of the pairs (p,x) and (q,y). Indeed,

$$\begin{aligned}P(X=x,\ Y=y)&=\frac{\binom{n}{x}\binom{n-x}{y}\cdot\binom{N-n}{p-x}\binom{(N-n)-(p-x)}{q-y}}{\binom{N}{p}\binom{N-p}{q}}\\&=\frac{\frac{n!}{x!y!(n-x-y)!}\cdot\frac{(N-n)!}{(p-x)!(q-y)!(N-n-(p-x)-(q-y))!}}{\frac{N!}{p!q!(N-p-q)!}},\end{aligned}$$

which is symmetric. Furthermore, the denominator denotes the number of ways to choose p and q elements from N elements, and so on.

The joint distribution is also called the (multivariate) hypergeometric distribution. The following contingency table would be helpful.

Table1 Contingency table

	with stones	without stones	total
eaten by pig	x	$p-x$	p
picked	y	$q-y$	q
rest	$n-x-y$	$(N-n)-(p-x)-(q-y)$	$N-p-q$
total	n	$N-n$	N

■**Remark 2: Standard error of the Monte Carlo estimate** In the above implementation of Monte Carlo, we have not calculated *the standard error*, which refers to "an estimate of the standard deviation of the Monte Carlo estimate". As we shall see, the standard error is also obtained with a little effort!

Let $\bar{X} = (1/N)\sum_{i=1}^{N} X_i$ be the Monte Carlo estimator. The standard deviation of $\bar{X}$ is given by

$$\sqrt{\mathrm{Var}[\bar{X}]} = \sqrt{\frac{1}{N}\mathrm{Var}[X_1]}.$$

The variance $\mathrm{Var}[X_1]$ is estimated by the sample variance

$$\hat{V} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2.$$

Therefore we obtain

$$(\text{standard error of } \bar{X}) = \sqrt{\frac{1}{N}\hat{V}}.$$

Let's check in our problem. Part (a) is straightforward:

Compute the standard error (a)

```
A.se = sqrt(var(Ys) / N)
```

The result was 0.0044. Thus the Monte Carlo estimate for (a) is expressed by *a confidence interval*

$$0.2543 \pm 0.0044.$$

The interval (fortunately) covers the true value 0.25, but this is not always the case. The probability that the interval covers the true value is about 68%, which is the probability $P(|Z| < 1)$ for the standard normal random variable Z. If you want to make the probability 95%, take

$$0.2543 \pm (1.96 \times 0.0044).$$

Here 1.96 is the number z such that $P(|Z| < z) = 0.95$. Anyway, the standard error gives us an ancillary information about the Monte Carlo estimate.

Part (b) is more challenging since we have to evaluate the error of $\bar{X}/\bar{Y}$ as an estimate of $E[X]/E[Y]$ for some random variables X and Y. This is evaluated by so-called *the Delta method.* Define random variables U_X and U_Y by $\bar{X} = E[X] + U_X$ and $\bar{Y} = E[Y] + U_Y$, respectively. Since U_X and U_Y are of $O(1/\sqrt{N})$, we have

$$\begin{aligned}\frac{\bar{X}}{\bar{Y}} &= \frac{E[X]+U_X}{E[Y]+U_Y} \\ &= \frac{E[X]}{E[Y]} + \frac{U_X}{E[Y]} - \frac{E[X]U_Y}{E[Y]^2} + O(N^{-1})\end{aligned}$$

by Taylor expansion. Therefore we have

$$\begin{aligned}\mathrm{Var}\left[\frac{\bar{X}}{\bar{Y}}\right] &\simeq \mathrm{Var}\left[\frac{U_X}{E[Y]} - \frac{E[X]U_Y}{E[Y]^2}\right] \\ &= \mathrm{Var}\left[\frac{\bar{X}}{E[Y]} - \frac{E[X]\bar{Y}}{E[Y]^2}\right] \\ &= \frac{1}{N}\mathrm{Var}\left[\frac{X}{E[Y]} - \frac{E[X]Y}{E[Y]^2}\right]\end{aligned}$$

Now we obtain

$$(\text{standard error of } \bar{X}/\bar{Y}) = \sqrt{\frac{1}{N}\hat{W}}, \quad \text{where} \quad \hat{W} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{X_i}{\bar{Y}} - \frac{\bar{X}Y_i}{\bar{Y}^2}\right)^2.$$

Let's check.

Compute the standard error (b)

```
Zs = (Xs > 0) & Ys
Ybar = mean(Ys)
Zbar = mean(Zs)
B.se = sqrt(var(Zs / Ybar - (Zbar/Ybar/Ybar) * Ys) / N)
```

The result was 0.0088. Thus the Monte Carlo estimate for (b) is expressed by the 95% confidence interval

$$0.7296 \pm (1.96 \times 0.0088).$$

The interval covers the true value 0.7417.

Problem 3.11.32

■The problem $N+1$ plates are laid out around a circular dining table, and a hot cake is passed between them in the manner of a symmetric random walk: each time it arrives on a plate, it is tossed to one of the two neighbouring plates, each possibility having probability $1/2$. The game stops at the moment when the cake has visited every plate at least once. Show that, with the exception of the plate where the cake began, each plate has probability $1/N$ of being the last plate visited by the cake.

■Answer Focus on the k-th plate $1 \le k \le N$. Let $T \ge 0$ be the time when the hot cake first arrives at either one of the neighbour of the k-th, where we interpret $T = 0$ if $k = 1$ or $k = N$. Note that T is finite with probability one (due to the absorbing probability). After the time T, by the Markov property and circular symmetry, the problem is reduced to the case of $k = 1$. In other words, the probability that the k-th plate is the last plate has to be common in all k. Therefore the probability is $1/N$.

■Checking by Monte Carlo Here is an R code for checking the answer. If you are not familiar with R language, just read the comments after the symbol #.

R code for checking Problem 3-11-32

```
LOOP = 1e4  # number of experiments
n = 7  # number of plates (except for the starting point)
freq = numeric(n)
for(Li in 1:LOOP){
  fill = logical(n)
  k = 0   # initial plate
  while(any(!fill)){
    x = sample(c(-1,1), 1)  # left or right
    k = (k + x) %% (n + 1)  # mod n+1
    if(k > 0) fill[k] = TRUE  # arrived at k
  }
  freq[k] = freq[k] + 1  # the last plate was k
}
p = freq / LOOP  # Monte Carlo estimate
p.sd = sqrt(p * (1-p) / LOOP)  # standard error
for(j in 1:n){
  cat(j, ":", p[j], "+-", p.sd[j], "\n")
}
cat("true:", 1 / n, "\n")
```

The output when $N = 7$ is

```
1 : 0.147 +- 0.003541059
2 : 0.1446 +- 0.003516971
3 : 0.1407 +- 0.003477118
4 : 0.1434 +- 0.003504803
5 : 0.1449 +- 0.00352
6 : 0.1418 +- 0.003488449
7 : 0.1376 +- 0.003444797
true: 0.1428571
```

2 Hints (or answers) for difficult problems

Problem 1.8.28

Note: In the problem, S should denote the sphere.

■Hint Set a cube C *randomly* such that all its vertices are on the surface. Let A_i $(i = 1, \dots, 8)$ be the event that the i-th vertex of C is red. It is enough to show that $P(A_1 \cap \cdots \cap A_8) > 0$ since then we can deduce that $A_1 \cap \cdots \cap A_8 \neq \emptyset$.

Problem 2.7.12

■Hint First try a simpler problem, that is, two coins rather than two dices.

■Answer Let $a_1, \dots, a_6$ be the probability that the first dice takes each value, and let $b_1, \dots, b_6$ be those for the second dice. The assumption that the sum of two numbers shown is equally likely is described as

$$\sum_i a_i b_{k-i} = \frac{1}{11}, \qquad 2 \le k \le 12,$$

where i runs over the set $\{i \mid 1 \le i \le 6,\ 1 \le k - i \le 6\}$. In particular, we have

$$a_1 b_1 = \frac{1}{11}, \quad a_6 b_6 = \frac{1}{11}, \quad a_1 b_6 + \cdots + a_6 b_1 = \frac{1}{11}$$

corresponding to $k = 2, 12, 7$. The first two equations imply $b_1 = 1/(11a_1)$ and $b_6 = 1/(11a_6)$. Then the last equation is

$$\frac{a_1}{11a_6} + \cdots + \frac{a_6}{11a_1} = \frac{1}{11}.$$

But this is impossible since

$$\frac{a_1}{11a_6} + \frac{a_6}{11a_1} \ge 2\sqrt{\frac{a_1}{11a_6} \cdot \frac{a_6}{11a_1}} = \frac{2}{11}.$$

Problem 2.7.13

Note: In the problem, the original statement $P(X = Y) = 1$ seems a mistake because there is no information here about joint distribution of X and Y.

■Hint For (a), first take arbitrary distributions F and G you like, and draw the graphs of $F(x)$, $G(x - \epsilon) - \epsilon$ and $G(x + \epsilon) + \epsilon$ as functions of x. After that, think logically. Part (b) is straightforward except for the last equality. For the last equality, prove the identity

$$|P(X \in A) - P(Y \in A)| = |P(X \notin A) - P(Y \notin A)|.$$

■Answer (a) (i) Non-negativity $d_{\mathrm{L}}(F,G) \geq 0$ is obvious from the definition. Symmetry $d_{\mathrm{L}}(F,G) = d_{\mathrm{L}}(G,F)$ follows from the equivalence

$$\begin{aligned}
&G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon \quad \text{for all } x \\
\iff\ &F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon \quad \text{for all } x
\end{aligned}$$

(ii) First assume $F = G$. Then $F(x) = G(x)$ for all x. For any $\epsilon > 0$, we have

$$F(x-\epsilon) - \epsilon < F(x) < F(x+\epsilon) + \epsilon$$

since F is nondecreasing. This implies $d_{\mathrm{L}}(F,F) = 0$. Conversely, suppose $d_{\mathrm{L}}(F,G) = 0$. This means

$$G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon \tag{$*$}$$

for any ϵ. By taking the limit as $\epsilon \downarrow 0$, we have

$$F(x) \leq G(x),$$

where the right continuity of G is used. By symmetry shown in (i), we also obtain

$$G(x) \leq F(x).$$

Therefore $F(x) = G(x)$ for all x.
(iii) Let F, G, H be distribution functions. Choose $a > d_{\mathrm{L}}(F,G)$ and $b > d_{\mathrm{L}}(G,H)$. Then, by definition, we have

$$G(x-a) - a \leq F(x) \leq G(x+a) + a, \quad H(x-b) - b \leq G(x) \leq H(x+b) + b$$

for all x. The two inequalities imply

$$H(x-a-b) - (a+b) \leq F(x) \leq H(x+a+b) + (a+b)$$

for all x. Thus $d_{\mathrm{L}}(F,H) \leq a+b$. By choosing a and b arbitrarily close to $d_{\mathrm{L}}(F,G)$ and $d_{\mathrm{L}}(G,H)$, respectively, we obtain $d_{\mathrm{L}}(F,H) \leq d_{\mathrm{L}}(F,G) + d_{\mathrm{L}}(G,H)$.
(b) (i) to (iii) are "clear" (= check by yourself). For the last equality, we note that for any $A \subset \mathbb{Z}$,

$$\begin{aligned}
d_{\mathrm{TV}}(X,Y) &= \sum_{x \in A} |P(X=x) - P(Y=x)| + \sum_{x \notin A} |P(X=x) - P(Y=x)| \\
&\geq \left| \sum_{x \in A} \{P(X=x) - P(Y=x)\} \right| + \left| \sum_{x \notin A} \{P(X=x) - P(Y=x)\} \right| \\
&= |P(X \in A) - P(Y \in A)| + |P(X \notin A) - P(Y \notin A)| \\
&= |P(X \in A) - P(Y \in A)| + |\{1 - P(X \in A)\} - \{1 - P(Y \in A)\}| \\
&= 2\,|P(X \in A) - P(Y \in A)|\,.
\end{aligned}$$

The equality holds if we put $A = \{x : P(X = x) > P(Y = x)\}$. Thus we obtain the desired result:

$$d_{\mathrm{TV}}(X, Y) = 2 \sup_{A \subset \mathbb{Z}} |P(X \in A) - P(Y \in A)| .$$

Problem 3.11.33

■Hint Derive a recurrence formula $r_j = \sum_{k=1}^{j-1} (j-1)^{-1}(1 + r_k)$ and solve it. The asymptotic expression may use Stirling's formula.

■Answer Consider the case that you are at the jth best and the next step is the kth best for a fixed $1 \le k \le j - 1$. This event occurs with probability $1/(j - 1)$. After the one step, you are at the kth best, and therefore the expected number of remaining steps you need is r_k. This explains a recurrence formula

$$r_j = \sum_{k=1}^{j-1} (j-1)^{-1}(1 + r_k).$$

The initial condition is $r_1 = 0$ by definition. Multiply $j - 1$ to obtain

$$(j-1)r_j = \sum_{k=1}^{j-1} (1 + r_k).$$

Taking difference with respect to j, we have

$$jr_{j+1} - (j-1)r_j = 1 + r_j,$$

or $r_{j+1} = r_j + 1/j$. Now it is easy to see

$$r_j = \frac{1}{j-1} + r_{j-1} = \sum_{k=1}^{j-1} \frac{1}{k} + r_1 = \sum_{k=1}^{j-1} \frac{1}{k}.$$

The expected time to reach B from the worst vertex is

$$\begin{aligned} r_{\binom{n}{m}} &= \sum_{k=1}^{\binom{n}{m}-1} \frac{1}{k} \\ &\asymp \log \binom{n}{m} \\ &= \log \frac{n!}{m!(n-m)!} \\ &\asymp n \log n \quad \text{(by Stirling's formula)} \end{aligned}$$

as long as $m \asymp n$, where $a_n \asymp b_n$ means a_n/b_n and b_n/a_n are bounded.

■Remark Note that the expected time is much smaller than "the worst time" $\binom{n}{m} - 1$.

Problem 3.11.34

■Hint Let f_n be the probability that m_1 remains isolated. Try to obtain a recurrence equation

$$f_n = \frac{1}{n-1}\sum_{k=1}^{n-1} f_{k-1}, \quad n \geq 2,$$

with $f_0 = 0$ and $f_1 = 1$. Solve this equation using generating functions, if necessary.

■Answer Let f_n be the probability that m_1 remains isolated. Consider the case that k-th pair of neighbours combined to form a stable dimer. This event occurs with probability $1/(n-1)$. After that, the problem is reduced to that for $n = k-1$ because kth molecule is combined with $(k+1)$th molecule and the first $(k-1)$ molecules remain unstable. Now we obtain the recursive formula

$$f_n = \frac{1}{n-1}\sum_{k=1}^{n-1} f_{k-1}, \quad n \geq 2,$$

with the initial condition $f_0 = 0$ and $f_1 = 1$. We solve this equation.

Multiply $n-1$ and replace k with $k+1$ to obtain

$$(n-1)f_n = \sum_{k=0}^{n-2} f_k, \quad n \geq 2.$$

Denote the generating function by $F(s) = \sum_{n=0}^{\infty} f_n s^n$. Since $\sum_{n=0}^{\infty} n f_n s^n = sF'(s)$ by term-by-term differentiation and $\sum_{n=0}^{\infty}(\sum_{k=0}^{n} f_k)s^n = F(s)/(1-s)$ by convolution, we have

$$sF'(s) - F(s) = s^2\frac{F(s)}{1-s}.$$

The initial condition is $F(0) = 0$ and $F'(0) = 1$. The differential equation is solved as follows:

$$\begin{aligned}
&F'(s) = \frac{1}{s}\left(1 + \frac{s^2}{1-s}\right)F(s),\\
\iff\ &\frac{F'(s)}{F(s)} = \frac{1}{s} + \frac{s}{1-s} = -1 + \frac{1}{s} + \frac{1}{1-s},\\
\iff\ &\log F(s) = C - s + \log|s| - \log|1-s|, \quad C\text{: arbitrary constant},\\
\iff\ &F(s) = \frac{Cse^{-s}}{1-s}, \quad C\text{: arbitrary constant},
\end{aligned}$$

By the initial condition, we obtain

$$F(s) = \frac{se^{-s}}{1-s}.$$

Finally, expand $F(s)$ as

$$\begin{aligned} F(s) &= s\sum_{k\geq 0} s^k \sum_{l\geq 0} \frac{(-s)^l}{l!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\frac{(-1)^l}{l!}\right)s^n. \end{aligned}$$

Now we obtain $f_n = \sum_{k=0}^{n-1}(-1)^k/k!$. Taylor's expansion implies $f_n \to e^{-1}$ as $n \to \infty$.

For the second part of the problem, let $g_n = E[U_n]$. As the first part, we obtain a recurrence formula

$$g_n = \frac{1}{n-1}\sum_{k=1}^{n-1}(g_{k-1} + g_{n-k-1}).$$

Indeed, if the kth pair of neighbours combined first, then $(k-1)$ consecutive molecules are unstable on the left and $(n-k-1)$ consecutive molecules are unstable on the right. The equation is rewritten as

$$(n-1)g_n = 2\sum_{k=0}^{n-2} g_k.$$

We also have the initial condition $g_0 = 0$ and $g_1 = 1$ from the definition. Solve this equation as above. Let $G(s) = \sum_{n=0}^{\infty} g_n s^n$. Then

$$\begin{aligned} & sG'(s) - G(s) = \frac{2s^2G(s)}{1-s} \\ \iff\ & \frac{G'(s)}{G(s)} = \frac{1}{s}\left(1 + \frac{2s^2}{1-s}\right) = \frac{1}{s} + \frac{2s}{1-s} = -2 + \frac{1}{s} + \frac{2}{1-s} \\ \iff\ & \log G(s) = C - 2s + \log|s| - 2\log|1-s| \\ \iff\ & G(s) = \frac{Cse^{-2s}}{(1-s)^2}. \end{aligned}$$

The initial condition implies

$$G(s) = \frac{se^{-2s}}{(1-s)^2}.$$

Expand to obtain

$$G(s) = s\sum_{k=1}^{\infty} ks^{k-1}\sum_{l=0}^{\infty}\frac{(-2s)^l}{l!} = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\frac{(n-l)(-2)^l}{l!}s^n.$$

Finally, we have

$$g_n = \sum_{k=0}^{n-1}\frac{(n-k)(-2)^k}{k!},$$

and therefore

$$\frac{g_n}{n} = \sum_{k=0}^{n-1}\frac{(-2)^k}{k!} - \frac{1}{n}\sum_{k=1}^{n-1}\frac{(-2)^k}{(k-1)!} \to e^{-2}$$

as $n \to \infty$.

Problem 3.9.1

■Hint The probability that the random walk is absorbed at 0 before N is given by

$$p_k = \frac{p^{n-K} q^k - q^N}{p^N - q^N}$$

(see p.74 of PRP). By using symmetry, derive the probability that the walk is absorbed at N before 0. Take their sum to prove $P(T < \infty) = 1$.

Next show that $P(T > N \mid S_0 = k) \leq 1 - q^N$ for any $0 \leq k \leq N$. Then by induction, one can show that

$$P(T > mN \mid S_0 = k) \leq (1 - q^N)^m$$

for all $m \geq 1$. This means the tail probability of T decays exponentially.

Note: There may be more elegant solutions. Please let me know if you find them.

Problem 3.9.2

■Hint The first claim pp_{k+1}/p_k is obtained by the definition of the conditional probability. Denote the duration time by T, which is a random variable. By the Markov property, you may obtain

$$\begin{aligned} E[T \mid W, S_0 = k] = P(S_1 = k+1 \mid W, S_0 = k)(1 + E[T \mid W, S_0 = k+1]) \\ + P(S_1 = k-1 \mid W, S_0 = k)(1 + E[T \mid W, S_0 = k-1]). \end{aligned}$$

Then the recurrence equation will follow.

If $p = \frac{1}{2}$, we have $p_k = 1 - \frac{k}{N}$ (p.74 of PRP). You can show that $\alpha_k := p_k J_k$ satisfies

$$\alpha_k = (1 - k/N) + \frac{1}{2}(\alpha_{k+1} + \alpha_{k-1})$$

with the boundary conditions $\alpha_0 = \alpha_N = 0$. Solve the equation by putting $\alpha_k = c_0 + c_1 k + c_2 k^2 + c_3 k^3$ for some constants c_0, c_1, c_2, c_3; or by using generating functions. The final solution will be $J_k = (2Nk)/3 - k^2/3$.

Problem 3.11.32

→ see p.5 of this document.

Problem 5.3.3

■Hint You may show that

$$p_{\mathrm{AA}}(2n) = \sum_{k=0}^{n} \frac{(2n)!}{(2k)!(2n-2k)!} \alpha^{2k} \beta^{2n-2k},$$

where $2k$ denotes the number of A-B or C-D steps in the $2n$ steps, and $2n - 2k$ denotes the number of A-C or B-D steps. Then prove

$$p_{\mathrm{AA}}(2n) = \frac{1}{2}\{(\alpha+\beta)^{2n} + (\alpha-\beta)^{2n}\}.$$

Now the generating function $G_{\mathrm{A}}(s)$ will be easily obtained. The generating function $F_{\mathrm{A}}(s)$ of the time of the first return to A is obtained by $G_{\mathrm{A}}(s) = 1 + F_{\mathrm{A}}(s)G_{\mathrm{A}}(s)$. See Theorem 1 in p.163 of PRP.

Problem 5.12.6

■Hint (a) Let (Z_n, W_n) be the n-th step, which takes $(1,0)$, $(-1,0)$, $(0,1)$ or $(0,-1)$ with probability $1/4$ each. Then $X_n = \sum_{k=1}^{n} Z_n$ and $Y_n = \sum_{k=1}^{n} W_n$. Use the properties of expectation.

(b) As the 1-dimensional case, the problem is reduce to calculate the generating function of $p_0(2n)$, which is the probability that the walk is at 0 after $2n$ steps. Denoting the number of upward steps by k, you will have

$$p_0(2n) = \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!}\left(\frac{1}{4}\right)^{2n}.$$

Use an identity related to the hypergeometric distribution:

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n}{n-k}}{\binom{2n}{n}} = 1,$$

where the summand of the left hand side is the probability that k red balls and $n-k$ white balls are drawn when n balls are randomly drawn from $2n$ balls consisting of n red balls and n white balls. Finally use Stirling's formula to evaluate the generating function.

Problem 5.12.10

■Hint Take it easy. To tell the truth, I have no answer to (b)–(d)... Let me know if you could solve!

■Answer (a) Let T be the duration of the game. Let $X_n = 1$ if the winner of the $(n-1)$-th game also wins the n-th game, and $X_n = 0$ otherwise. We set $X_1 = 1$ identically. Note that X_n, $n \geq 2$, are independent and $P(X_n = 1) = \frac{1}{2}$. Now T is represented by

$$T = \min\{n \geq r-1 \mid X_n = \cdots = X_{n-r+2} = 1\}.$$

Classify the first consecutive wins to obtain

$$E[s^T] = \sum_{k=2}^{r-1} P(X_2 = 1, \ldots, X_{k-1} = 1, X_k = 0)E[s^{k-1+T}] + P(X_2 = \cdots = X_{r-1} = 1)s^{r-1},$$

and therefore the generating function $G(s) = E[s^T]$ satisfies

$$G(s) = \sum_{k=2}^{r-1} \frac{s^{k-1}}{2^{k-1}} G(s) + \frac{s^{r-1}}{2^{r-2}}.$$

We obtain

$$G(s) = \frac{s^{r-1}/2^{r-2}}{1 - \sum_{k=2}^{r-1} s^{k-1}/2^{k-1}} = \frac{s^{r-1}(2-s)}{2^{r-1}(1-s) + s^{r-1}}.$$

In particular,

$$G'(1) = 2^{r-1} - 1.$$

■Remark If $r = 3$, the game is the same as "Tomoe sen" in Sumo wrestling. See `https://ja.wikipedia.org/wiki/巴戦`.