Theory of Stochastic Processes 2017 S1S2, Midterm Exam

May 27

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- This examination paper consists of 5 questions including an optional question.
- This is an open-book and open-note examination.
- Write your answers in English in the answer sheets provided. Exception: write your name in Japanese characters (if you have).
- The time allowed is 90 minutes.

Q 1. Let $0 < p < 1$ and $q = 1 - p$. Define a Markov chain $S = \{S_n\}_{n \geq 0}$ taking values in $\{0, 1, \cdots\}$ by

$$P(S_n = j \mid S_{n-1} = i) = \begin{cases} p & \text{if } i \geq 1 \text{ and } j = i+1, \\ q & \text{if } i \geq 1 \text{ and } j = i-1, \\ 1 & \text{if } i = 0 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, S is a simple random walk with the reflecting boundary at 0.

(a) Calculate $P(S_4 = 0 \mid S_0 = 0)$. (10 marks)

(b) Let $0 < p < 1/2$. Find a stationary distribution of the chain. (10 marks)

(c) Let $1/2 < p < 1$. Prove that the state 0 is transient. (15 marks)

Q 2. Let a, b, c be positive numbers. Define a continuous-time Markov chain $\{X(t)\}_{t \geq 0}$ by the generator

$$\boldsymbol{G} = \begin{pmatrix} -(a+b) & a & b \\ a & -(a+c) & c \\ b & c & -(b+c) \end{pmatrix}.$$

(a) Find a stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ of the chain. (10 marks)

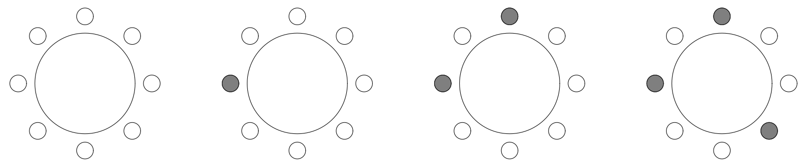
(b) Suppose that $a = b = c$. Find the transition matrix $\boldsymbol{P}_t = (p_{ij}(t))$, where

$$p_{ij}(t) = P(X(t) = j \mid X(0) = i).$$

(15 marks)

Q 3. Throw a dice once and let Z_1 be the number that turned up. Then throw the dice Z_1 times and let Z_2 be the sum of the numbers that turned up. Similarly, after Z_{n-1} is defined, throw the dice Z_{n-1} times and let Z_n be the sum of the numbers that turned up. Find the expected value of Z_n for each n. (15 marks)

Q 4. There are n seats at a table. At each time, a person chooses an empty seat at random in such a way that both seats next to it are also empty. This process continues until there are no seat available. Then let X_n be the number of the occupied seats. For example, the following figure shows an outcome when $n = 8$.



8 available | 5 available | 3 available | no available $X_n = 3$.

(a) Find the expected value of X_6. (10 marks)

(b) Find the expected value of X_n for every n. (15 marks)

Q 5 (Optional question)**.** Please answer your research interest, regardless of relation to the course. Describe any comments or suggestions about the course. (up to 10 marks)