Theory of Stochastic Processes 2017 S1S2, Midterm Exam (Answer)

June 8, 2017

Tomonari SEI

Q1

Let $0 < p < 1$ and $q = 1 - p$. Define a Markov chain $S = \{S_n\}_{n\geq 0}$ taking values in $\{0, 1, \cdots\}$ by

$$P(S_n = j \mid S_{n-1} = i) = \begin{cases} p & \text{if } i \geq 1 \text{ and } j = i+1, \\ q & \text{if } i \geq 1 \text{ and } j = i-1, \\ 1 & \text{if } i = 0 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, S is a simple random walk with the reflecting boundary at 0.

(a) Calculate $P(S_4 = 0 \mid S_0 = 0)$. (10 marks)

(b) Let $0 < p < 1/2$. Find a stationary distribution of the chain. (10 marks)

(c) Let $1/2 < p < 1$. Prove that the state 0 is transient. (15 marks)

Answer. (a) Possible paths are

$$(S_0, S_1, S_2, S_3, S_4) = (0, 1, 2, 1, 0) \quad \text{or} \quad (0, 1, 0, 1, 0).$$

They have probability $1 \cdot p \cdot q \cdot q = pq^2$ and $1 \cdot q \cdot 1 \cdot q = q^2$, respectively. Therefore

$$P(S_4 = 0 | S_0 = 1) = pq^2 + q^2.$$

(b) The equation which determines the stationary distribution is

$$\pi_0 = q\pi_1, \tag{1}$$
$$\pi_1 = \pi_0 + q\pi_2 \quad \text{and} \tag{2}$$
$$\pi_i = p\pi_{i-1} + q\pi_{i+1}, \quad i \geq 2. \tag{3}$$

The general solution for the recursive equation (3) is

$$\pi_i = a + b(p/q)^i, \quad i \geq 1,$$

since the characteristic equation of (3) is $\lambda = p + q\lambda^2$ and its solutions are $\lambda = 1$ and $\lambda = p/q$. Since π_i is a mass function, it should satisfy

$$\sum_{i\geq 0} \pi_i = 1 \tag{4}$$

and therefore the coefficient a must be 0. Now we obtain $\pi_i = b(p/q)^i$. Then π_0 is determined by (1):

$$\pi_0 = q\pi_1 = bq(p/q) = bp.$$

Finally, the coefficient b is determined by (4):

$$\sum_{i\geq 0} \pi_i = bp + \sum_{i\geq 1} b(p/q)^i = bp + \frac{b(p/q)}{1 - p/q} = b\frac{p(1 - p/q + 1/q)}{1 - p/q} = \frac{2bpq}{q - p} = 1,$$

and therefore $b = (q-p)/(2pq)$. The stationary distribution is

$$\pi_0 = \frac{q-p}{2q} \quad \text{and} \quad \pi_i = \frac{q-p}{2pq}(p/q)^i \quad (i \geq 1).$$

(c) **(Solution 1: counting paths)** Let $f_0(n) = P(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0 | S_0 = 0)$ for $n \geq 1$ and $f_0(0) = 0$. If $\sum_{n\geq 0} f_0(n) < 1$, the chain is transient. It is easy to see that $f_0(n) = 0$ for odd n's. Consider $f_0(2n)$ for $n \geq 1$. Let A_{2n} be the number of paths satisfying $S_0 = 0, S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 0$. This is obtained by reflection principle as follows. The number of paths with $S_1 = 1$ and $S_{2n-1} = 1$ is $\binom{2n-2}{n-1}$, and the number of paths with $S_1 = 1$, $S_{2n-1} = 1$ and $S_k = 0$ for some $2 \leq k \leq 2n-2$ is $\binom{2n-2}{n}$ by the reflection principle. Therefore

$$A_{2n} = \binom{2n-2}{n-1} - \binom{2n-2}{n} = \frac{(2n-2)!}{(n-1)!} - \frac{(2n-2)!}{n!(n-2)!} = \frac{(2n-2)!}{n!(n-1)!}.$$

Since each path has the probability $p^{n-1}q^n$, we have

$$f_0(2n) = A_{2n}p^{n-1}q^n = \frac{(2n-2)!}{n!(n-1)!}p^{n-1}q^n.$$

Their sum is

$$\begin{aligned}
\sum_{n=1}^{\infty} f_0(2n) &= \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!}p^{n-1}q^n \\
&= \sum_{n=1}^{\infty} \frac{(n-3/2)\cdots(1/2)}{n!}2^{2n-2}p^{n-1}q^n \\
&= \sum_{n=1}^{\infty} \frac{(1/2)(-1/2)\cdots(-(n-3/2))}{n!}(-1)^{n-1}2^{2n-1}p^{n-1}q^n \\
&= \frac{-1}{2p}\sum_{n=1}^{\infty} \frac{(1/2)(-1/2)\cdots(-(n-3/2))}{n!}(-4pq)^n \\
&= \frac{1}{2p}\left\{1 - \sqrt{1-4pq}\right\} \\
&= \frac{1}{2p}(1 - |1-2p|) \\
&= \begin{cases} 1 & \text{if } 0 < p \leq 1/2, \\ (1-p)/p & \text{if } 1/2 < p < 1. \end{cases}
\end{aligned}$$

The chain is transient if (and only if) $1/2 < p < 1$.

(Solution 2: using absorbing barrier) Fix $b \geq 1$ and consider an event

$$A = A(b) = \{S_0 < b, \ldots, S_{n-1} < b, S_n = 0 \text{ for some } n \geq 0\}.$$

In other words, the event A denotes that the chain reaches the state 0 before reaching the state b. Let $\phi_j = \phi_j(b) = P(A|S_0 = j)$ for each $0 \leq j \leq b$. Then we have a recursive equation

$$\phi_j = p\phi_{j+1} + q\phi_{j-1} \quad (1 \leq j \leq b-1)$$

with boundary conditions $\phi_b = 0$ and $\phi_0 = 1$. The unique solution is

$$\phi_j = \frac{(q/p)^j - (q/p)^b}{1-(q/p)^b}.$$

By letting $b \to \infty$, we have

$$\lim_{b\to\infty} \phi_j(b) = (q/p)^j$$

since $q/p < 1$. Since the event $A(b)$ is increasing in b, we have

$$\begin{aligned} P(S_n = 0 \text{ for some } n \geq 0 | S_0 = j) &= P\left(\bigcup_{b\geq 1} A(b) \middle| S_0 = j \right) \\ &= \lim_{b\to\infty} P(A(b)|S_0 = j) \\ &= \lim_{b\to\infty} \phi_j(b) \\ &= (q/p)^j. \end{aligned}$$

In particular,

$$P(S_n = 0 \text{ for some } n \geq 1 | S_0 = 0) = P(S_n = 0 \text{ for some } n \geq 1 | S_1 = 1) = q/p < 1.$$

Note that this result is the same as obtained in Solution 1. Hence S is transient.

(Solution 3: using Theorem 6.4.10 of the book PRP) Consider a system of equations

$$y_i = py_{i+1} + qy_{i-1}, \quad i \geq 1, \quad y_0 = 0. \tag{5}$$

The general solution is

$$y_i = a - a(q/p)^i, \quad i \geq 1,$$

where a is any real number. In particular, $|y_j| \leq 1$ for all $j \geq 1$ whenever $|a| \leq 1$. Hence the chain is transient by Theorem 6.4.10. Note that the equation (5) is satisfied by $y_i = \phi_i - 1$, where $\phi_i = P(S_n = 0 \text{ for some } n \geq 0 | S_0 = i)$.

(Solution 4: comparison) Let $Y = \{Y_n\}_{n=0}^{\infty}$ be the simple random walk in the usual sense. Let

$$T = \inf\{n \geq 1 \mid S_n = 0\} \quad \text{and} \quad U = \inf\{n \geq 1 \mid Y_n = 0\}.$$

Then we have $P(T = 2n|S_0 = 0) \leq P(U = 2n|Y_0 = 0)$. Indeed, as shown in the following figure, a specific path $\{s_i\}_{i=0}^{2n}$ of S has the probability $p^{n-1}q^n$ whereas the corresponding two paths $\{s_i\}$ and $\{-s_i\}$ of Y have the probability $2p^nq^n$, which is greater than $p^{n-1}q^n$. Since Y is transient, we have $P(T < \infty|S_0 = 0) \leq P(U < \infty|Y_0 = 0) < 1$. Hence S_n is also transient.

A path $\{s_i\}_{i=0}^{2n}$ of S. Two paths $\{s_i\}_{i=0}^{2n}$ and $\{-s_i\}_{i=0}^{2n}$ of Y.

$\square$

Q2

Let a, b, c be positive numbers. Define a continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ by the generator

$$\boldsymbol{G} = \begin{pmatrix} -(a+b) & a & b \\ a & -(a+c) & c \\ b & c & -(b+c) \end{pmatrix}.$$

(a) Find a stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ of the chain. (10 marks)

(b) Suppose that $a = b = c$. Find the transition matrix $\boldsymbol{P}_t = (p_{ij}(t))$, where

$$p_{ij}(t) = P(X(t) = j \mid X(0) = i).$$

(15 marks)

Answer. (a) Solve the linear equation $\boldsymbol{\pi G} = \boldsymbol{0}$ with $\pi_1 + \pi_2 + \pi_3 = 1$. The equations are

$$\begin{aligned} -(a+b)\pi_1 + a\pi_2 + b\pi_3 &= 0, \\ a\pi_1 - (a+c)\pi_2 + c\pi_3 &= 0, \\ b\pi_1 + c\pi_2 - (b+c)\pi_3 &= 0, \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

Note that a, b, c are positive. By eliminating π_3 from the first two equations, we have

$$\{-(a+b) - (b/c)a\}\pi_1 + \{a + (a+c)b/c\}\pi_2 = 0.$$

We obtain $\pi_1 = \pi_2$. Similarly, we have $\pi_2 = \pi_3$. Therefore $\pi_1 = \pi_2 = \pi_3 = 1/3$.

(Remark) In general, if the state space is finite and the generator $\boldsymbol{G}$ is symmetric, then the uniform distribution $\pi_i = 1/|S|$ is a stationary distribution, which may not be unique.

(b) The generator is

$$\boldsymbol{G} = \begin{pmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{pmatrix}.$$

The transition matrix $\boldsymbol{P}_t$ is obtained by $\boldsymbol{P}_t = \exp(t\boldsymbol{G})$. The spectral decomposition of $\boldsymbol{G}$ is

$$\boldsymbol{G} = \boldsymbol{V\Lambda V}^\top, \quad \boldsymbol{V} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2\sqrt{6} \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} 0 & & \\ & -3a & \\ & & -3a \end{pmatrix}.$$

Therefore

$$\begin{aligned} \boldsymbol{P}_t &= \exp(t\boldsymbol{G}) \\ &= \boldsymbol{V} \exp(t\boldsymbol{\Lambda}) \boldsymbol{V}^\top \\ &= \boldsymbol{V} \begin{pmatrix} 1 & & \\ & e^{-3at} & \\ & & e^{-3at} \end{pmatrix} \boldsymbol{V}^\top \\ &= \begin{pmatrix} (1/3) + (2/3)e^{-3at} & (1/3) - (1/3)e^{-3at} & (1/3) - (1/3)e^{-3at} \\ (1/3) - (1/3)e^{-3at} & (1/3) + (2/3)e^{-3at} & (1/3) - (1/3)e^{-3at} \\ (1/3) - (1/3)e^{-3at} & (1/3) - (1/3)e^{-3at} & (1/3) + (2/3)e^{-3at} \end{pmatrix}. \end{aligned}$$

□

Q3

Throw a dice once and let Z_1 be the number that turned up. Then throw the dice Z_1 times and let Z_2 be the sum of the numbers that turned up. Similarly, after Z_{n-1} is defined, throw the dice Z_{n-1} times and let Z_n be the sum of the numbers that turned up. Find the expected value of Z_n for each n.

(15 marks)

Answer. **(Solution 1)** Let $\{X_i^n\}_{i\geq 1, n\geq 1}$ be independent random variables with the probability $P(X_1^n = j) = 1/6$ for each $j = 1, \dots, 6$. The expected value of X_i^n is $E[X_i^n] = (1 + \cdots + 6)/6 = 7/2$. Then Z_n is written as

$$Z_n = X_1^n + \cdots + X_{Z_{n-1}}^n.$$

The expected value is

$$\begin{aligned} E[Z_n] &= E[X_1^n + \cdots + X_{Z_{n-1}}^n] \\ &= E[E[X_1^n + \cdots + X_{Z_{n-1}}^n | Z_{n-1}]] \\ &= E[(7/2)Z_{n-1}] \\ &= (7/2)E[Z_{n-1}]. \end{aligned}$$

Since $Z_0 = 1$, we have $E[Z_n] = (7/2)^n$.

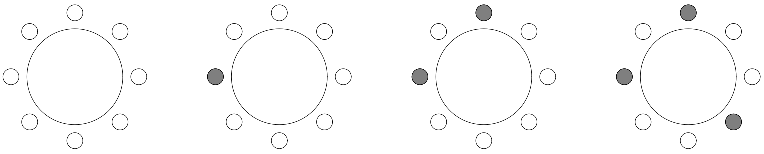
(Solution 2) The process Z_n is a branching process with the family size distribution $f(k) = 1/6$ for $k = 1, \dots, 6$. The generating function $G_n(s)$ of Z_n satisfies $G_n(s) = G_{n-1}(G(s)) = G(G_{n-1}(s))$. The expected value of Z_n is $G_n'(1)$. In general, we have

$$\begin{aligned} G_n'(1) &= G'(G_{n-1}(1))G_{n-1}'(1) \\ &= G'(1)G_{n-1}'(1) \\ &= (G'(1))^n \end{aligned}$$

(see Lemma 5.4.2 of PRP). Since $G'(1) = 7/2$, we have $G_n'(1) = (7/2)^n$. □

Q4

There are n seats at a table. At each time, a person chooses an empty seat at random in such a way that both seats next to it are also empty. This process continues until there are no seat available. Then let X_n be the number of the occupied seats. For example, the following figure shows an outcome when $n = 8$.



8 available 5 available 3 available no available

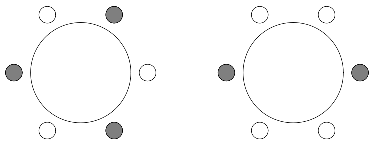
$X_n = 3$.

(a) Find the expected value of X_6. (10 marks)

(b) Find the expected value of X_n for every n. (15 marks)

Answer. (a) If $n = 6$, then possible outcomes are (i) $X_6 = 3$ with probability $2/3$, or (ii) $X_6 = 2$ with probability $1/3$ (see the following figure). Therefore the expected value of X_6 is

$$E[X_6] = 3 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{8}{3}.$$

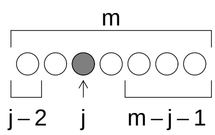


(i) (ii)

(b) After a seat is selected, there remain $n-3$ seats available in line. Let Y_m be the number of eventually occupied seats out of the m seats in line. Then we have $E[X_n] = 1 + E[Y_{n-3}]$. It is sufficient to calculate the expected value $\mu_m = E[Y_m]$. By conditioning the first selected seat in the m seats, we have the following recursive formula:

$$\mu_m = \frac{1}{m} \sum_{j=1}^{m} (1 + \mu_{j-2} + \mu_{m-j-1}), \quad m \geq 1,$$

where $\mu_{-1} = \mu_0 = 0$. See the following figure.



It is further rewritten as

$$m\mu_m = m + 2\sum_{j=1}^{m} \mu_{j-2}, \quad m \geq 1.$$

Denote the generating function of μ_m by $G(s) = \sum_{m=0}^{\infty} \mu_m s^m$. Multiplying s^m to the above equation and summing up, we have

$$\sum_{m=1}^{\infty} m\mu_m s^m = \sum_{m=1}^{\infty} \left(m + 2\sum_{j=1}^{m} \mu_{j-2} \right) s^m.$$

Since $G'(s) = \sum_{m=1}^{\infty} m\mu_m s^{m-1}$, we obtain

$$\begin{aligned}
sG'(s) &= \sum_{m=1}^{\infty} ms^m + 2\sum_{m=3}^{\infty}\sum_{k=1}^{m-2} \mu_k s^m \\
&= s\left(\frac{1}{1-s}\right)' + 2\sum_{k=1}^{\infty} \mu_k \sum_{m=k+2}^{\infty} s^m \\
&= \frac{s}{(1-s)^2} + 2\sum_{k=1}^{\infty} \mu_k \frac{s^{k+2}}{1-s} \\
&= \frac{s}{(1-s)^2} + \frac{2s^2}{1-s} G(s).
\end{aligned}$$

We obtain the following differential equation

$$\begin{aligned}
G'(s) &= \frac{1}{(1-s)^2} + \frac{2s}{1-s} G(s) \\
&= \frac{1}{(1-s)^2} + \left(-2 + \frac{2}{1-s}\right) G(s)
\end{aligned}$$

The equation is equivalent to

$$G'(s) + \left(2 - \frac{2}{1-s}\right) G(s) = \frac{1}{(1-s)^2}.$$

Since $\int (2 - \frac{2}{1-s}) ds = 2s + 2\log(1-s) = \log((1-s)^2 e^{2s})$, we have

$$\left((1-s)^2 e^{2s} G(s)\right)' = e^{2s}.$$

The general solution is

$$G(s) = \frac{1 + Ce^{-2s}}{2(1-s)^2},$$

where C is an arbitrary constant. Since $G(0) = 0$, the constant is $C = -1$. Expand $G(s)$ as

$$\begin{aligned}
G(s) &= \frac{1 - e^{-2s}}{2(1-s)^2} \\
&= \frac{1}{2}\sum_{i=0}^{\infty} (i+1)s^i \sum_{j=1}^{\infty} \frac{(-1)^{j-1}(2s)^j}{j!} \\
&= \frac{1}{2}\sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{(m-j+1)(-1)^{j-1}2^j}{j!} \right) s^m
\end{aligned}$$

Therefore

$$\mu_m = \frac{1}{2}\sum_{j=1}^{m} \frac{(m-j+1)(-1)^{j-1}2^j}{j!}.$$

Finally, the expected value of X_n is

$$E[X_n] = 1 + E[Y_{n-3}] = 1 + \frac{1}{2}\sum_{j=1}^{n-3} \frac{(n-2-j)(-1)^{j-1}2^j}{j!}.$$

(Remark) Compare the obtained formula with direct calculation. For example,

$$E[X_6] = 1 + \frac{1}{2}\left(\frac{3\cdot 2}{1} - \frac{2\cdot 4}{2} + \frac{1\cdot 8}{6}\right) = \frac{8}{3}$$

is consistent with the result of (a). The following table shows $E[X_n]$ for $1 \le n \le 10$ according to the formula.

n	1	2	3	4	5	6	7	8	9	10
$E[X_n]$	1	1	1	2	2	$\frac{8}{3}$	3	$\frac{52}{15}$	$\frac{35}{9}$	$\frac{454}{105}$

$\square$