Theory of Stochastic Processes

Lecture 13: Review (partially solved problems)

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Solutions to a part of recommended problems are given.

Lecture 8: Markov chain Monte Carlo

■Problem 6.5.1 By induction, we obtain

$$\pi_i = \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i} \pi_0.$$

Then

$$\pi_i p_{i,i+1} = \frac{\lambda_0 \cdots \lambda_i}{\mu_1 \cdots \mu_i} \pi_0 = \pi_{i+1} p_{i+1,i}, \quad i = 0, \ldots, b-1.$$

For other pairs (i, j), $i \neq j$, $\pi_i p_{ij} = 0 = \pi_j p_{ji}$. Therefore the chain is reversible.

■Problem 6.5.8 Suppose $\boldsymbol{P}$ is reversible. Let $\boldsymbol{\pi}$ be the stationary distribution and $\boldsymbol{D}_{\boldsymbol{\pi}}$ be the diagonal matrix with diagonal entries $\boldsymbol{\pi}$. Then $\boldsymbol{S} = \boldsymbol{D}_{\boldsymbol{\pi}} \boldsymbol{P}$ is a symmetric matrix by definition of reversibility. We obtain $\boldsymbol{P} = \boldsymbol{D}_{\boldsymbol{\pi}}^{-1} \boldsymbol{S}$.

Conversely, suppose $\boldsymbol{P} = \boldsymbol{D}\boldsymbol{S} = (d_i S_{ij})$. Define $\pi_i = (1/d_i)/\sum_k (1/d_k)$. Then $\pi_i p_{ij} = \pi_i d_i S_{ij} = S_{ij}/\sum_k (1/d_k)$ is symmetric. Therefore $\boldsymbol{P}$ is reversible.

Next suppose $\boldsymbol{P} = \boldsymbol{D}\boldsymbol{S}$. Let $\boldsymbol{T} = \boldsymbol{D}^{-1/2} \boldsymbol{P} \boldsymbol{D}^{1/2}$. Then $\boldsymbol{T}$ has the same eigenvalues as $\boldsymbol{P}$. Since $\boldsymbol{T} = \boldsymbol{D}^{1/2} \boldsymbol{S} \boldsymbol{D}^{1/2}$ is symmetric, the eigenvalues are real.

Finally, consider a transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} + \alpha & \frac{1}{3} - \alpha \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} - \alpha & \frac{1}{3} + \alpha \end{pmatrix}, \quad \alpha \in (-\tfrac{1}{3}, \tfrac{1}{3}).$$

The stationary distribution is $\boldsymbol{\pi} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The chain is not reversible unless $\alpha = 0$. On the other hand, $\boldsymbol{P}$ has the real eigenvalues $1, \alpha, 0$.

Lecture 9: Stationary processes

■**Problem 2** The process is represented as

$$X_n = g(L)Z_n, \quad g(z) = \sigma(1 - \alpha_1 z - \cdots - \alpha_p z^p)^{-1}.$$

Since all the roots of $1 - \alpha_1 z - \cdots - \alpha_p z^p = 0$ are outside of the unit circle, $g(z)$ has the convergence radius greater than 1. Then the spectral density is given by

$$f(\lambda) = \frac{1}{2\pi}|g(e^{i\lambda})|^2 = \frac{1}{2\pi|1 - \sum_{j=1}^p \alpha_i e^{ij\lambda}|^2}.$$

by Theorem 2 of the lecture note of Lecture 9.

■**Problem 4** (i) Since A and B have the mean zero, the mean of X_n is

$$E[X_n] = E[A\cos(\Omega n) + B\sin(\Omega n)] = 0.$$

Since $E[A^2] = E[B^2] = 1$ and $E[AB] = 0$, the covariance of X_n and X_{n+m} is

$$\begin{aligned} E[X_n X_{n+m}] &= E[\cos(\Omega n)\cos(\Omega(n+m))] + E[\sin(\Omega n)\sin(\Omega(n+m))] \\ &= \frac{1}{\pi}\int_0^\pi (\cos(\omega n)\cos(\omega(n+m)) + \sin(\omega n)\sin(\omega(n+m)))\,d\omega \\ &= \frac{1}{\pi}\int_0^\pi \cos(\omega m)d\omega = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore X_n is the white noise. In particular, it is stationary.

(ii) We have $X_0 = A$, $X_1 + X_{-1} = 2A\cos(\Omega)$, and $X_1 - X_{-1} = 2B\sin(\Omega)$. The first two equations determine A and Ω. If $\Omega \neq 0$ and $\Omega \neq \pi$, then the third equation determines B and therefore all X_n are determined. Even if $\Omega = 0$ or $\Omega = \pi$, $X_n = A\cos(\Omega n)$ is determined.

■**Problem 5** For real numbers $v_1, \ldots, v_N$,

$$\sum_{j=1}^N \sum_{k=1}^N v_j v_k c(j-k) = \sum_{j=1}^N \sum_{k=1}^N v_j v_k E[X_j X_k] = E\left[\left(\textstyle\sum_{j=1}^N v_j X_j\right)^2\right] \geq 0.$$

Hence the matrix $\boldsymbol{C} = \{c(j-k)\}$ is non-negative definite[*1]. All the eigenvectors of $\boldsymbol{C}$ are given by $\boldsymbol{v}_m = (N^{-1/2}e^{2\pi imn/N})_{n=1}^N$ for $m = 0, \ldots, N-1$. Indeed,

$$\begin{aligned} (\boldsymbol{C}\boldsymbol{v}_m)_k &= N^{-1/2}\sum_{n=1}^N c(k-n)e^{2\pi imn/N} \\ &= N^{-1/2}e^{2\pi imk/N}\sum_{j=1}^N c(-j)e^{2\pi imj/N}, \quad j = n-k, \\ &= Nf(m)(\boldsymbol{v}_m)_k, \end{aligned}$$

[*1] The assumption $c(n) = c(n-N)$ is not necessary here.

where $f(m) = N^{-1}\sum_{n=0}^{N-1} c(n)e^{-2\pi imn/N}$. The spectral decomposition of $\boldsymbol{C}$ is $\boldsymbol{C} = \sum_{m=0}^{N-1} Nf(m)\boldsymbol{v}_m\boldsymbol{v}_m^*$ and therefore

$$c(j-k) = \sum_{m=0}^{N-1} Nf(m)(\boldsymbol{v}_m)_j(\boldsymbol{v}_m^*)_k = \sum_{m=0}^{N-1} f(m)e^{2\pi im(j-k)/N}.$$

■Problem 6 (i) It is easy to see that $E[\bar{X}] = N^{-1}\sum_n E[X_n] = \mu$. Furthermore,

$$\begin{aligned}
V[\bar{X}] = E[(\bar{X}-\mu)^2] &= N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} E[(X_j-\mu)(X_k-\mu)] \\
&= N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} \sigma^2\rho(j-k) \\
&= N^{-2}\sum_{n=-(N-1)}^{N-1}\sum_{(j,k):j-k=n} \sigma^2\rho(n) \\
&= \sigma^2 N^{-1}\sum_{n=-(N-1)}^{N-1} (1-|n|/N)\rho(n).
\end{aligned}$$

(ii)

$$\begin{aligned}
E[\hat{\sigma}^2] = N^{-1}\sum_{n=1}^{N} E[(X_n-\bar{X})^2] &= N^{-1}\sum_{n=1}^{N} E[((X_n-\mu)-(\bar{X}-\mu))^2] \\
&= N^{-1}\sum_{n=1}^{N}\left\{E[(X_n-\mu)^2] - 2E[(X_n-\mu)(\bar{X}-\mu)] + E[(\bar{X}-\mu)^2]\right\} \\
&= N^{-1}\left\{N\sigma^2 - 2NE[(\bar{X}-\mu)^2] + NE[(\bar{X}-\mu)^2]\right\} = \sigma^2 - V[\bar{X}].
\end{aligned}$$

(iii) By (i) and Abel's theorem (or monotone convergence theorem), we have

$$\lim_{N\to\infty} NV[\bar{X}] = \lim_{N\to\infty}\sum_{n=-(N-1)}^{N-1}(1-|n|/N)\rho(n) = \sum_{n=-\infty}^{\infty}\rho(n) = f(0).$$

In particular, $\lim_{N\to\infty} V[\bar{X}] = 0$. By (ii), we also have

$$\lim_{N\to\infty} E[\hat{\sigma}^2] = \lim_{N\to\infty}(\sigma^2 - V[\bar{X}]) = \sigma^2.$$

Lecture 10: Martingales

■**Problem 12.1.3** $E[Z_{n+1}\mu^{-n-1}|\mathcal{F}_n] = E[Z_{n+1}|Z_n]\mu^{-n-1} = (\mu Z_n)\mu^{-n-1} = Z_n\mu^{-n}$.
$E[\eta^{Z_{n+1}}|\mathcal{F}_n] = E[\eta^{Z_{n+1}}|Z_n] = G(\eta)^{Z_n} = \eta^{Z_n}$, where G is the generating function of Z_1

■**Problem 12.1.5** $E[(Y_k - Y_j)Y_j] = E[E[Y_k - Y_j|\mathcal{F}_j]Y_j] = E[(Y_j - Y_j)Y_j] = 0$.
$E[(Y_k - Y_j)^2|\mathcal{F}_i] = E[Y_k^2|\mathcal{F}_i] - 2E[E[Y_k|\mathcal{F}_j]Y_j|\mathcal{F}_i] + E[Y_j^2|\mathcal{F}_i] = E[Y_k^2|\mathcal{F}_i] - E[Y_j^2|\mathcal{F}_i]$.
Suppose that $E[Y_n^2] \le K$ for all n. Since $E[(Y_n - Y_m)^2] = E[Y_n^2] - E[Y_m^2]$ for $m < n$, $E[Y_n^2]$ is non-decreasing. Therefore there exists $\lim_{n\to\infty} E[Y_n^2] \le K$. In particular, for any $\epsilon > 0$, there exists n_0 such that $|E[Y_n^2] - E[Y_m^2]| < \epsilon$ for all $n, m \ge n_0$. Then $E[(Y_n - Y_m)^2] = E[Y_n^2] - E[Y_m^2] < \epsilon$. Therefore Y_n is a Cauchy sequence and converges in mean square*2.

■**Problem 12.1.6** By Jensen's inequality, we have

$$E[u(Y_{n+1})|\mathcal{F}_n] \ge u(E[Y_{n+1}|\mathcal{F}_n]) = u(Y_n).$$

This means $u(Y_n)$ is a submartingale. The processes $|Y_n|$, Y_n^2, Y_n^+ are submartingales since functions $u(y) = |y|, y^2, y^+ = \max(0, y)$ are convex, respectively.

■**Problem 12.2.1** We apply McDiarmid's inequality. Let $f((v_1, w_1), \dots, (v_n, w_n))$ be the maximum worth when the volume and worth of the i-th object are v_i and w_i. If an object (v_i, w_i) is replaced with (v_i', w_i'), then f changes at most M. Indeed, after the i-th object is removed if necessary, the total cost is reduced at most M while the packed objects are feasible. Therefore

$$f((v_1, w_1), \dots, (v_i, w_i), \dots, (v_n, w_n)) - M \le f((v_1, w_1), \dots, (v_i', w_i'), \dots, (v_n, w_n))$$

and

$$f((v_1, w_1), \dots, (v_i', w_i'), \dots, (v_n, w_n)) - M \le f((v_1, w_1), \dots, (v_i, w_i), \dots, (v_n, w_n)).$$

Now, McDiarmid's inequality implies that

$$P(|Z - E[Z]| \ge nM\varepsilon) \le 2\exp(-n\varepsilon^2/2), \quad \varepsilon > 0.$$

Let $x = nM\varepsilon$ to obtain

$$P(|Z - E[Z]| \ge x) \le 2\exp(-x^2/(2nM^2)), \quad x > 0.$$

*2 Here we used the completeness property of L^2. This was beyond the scope of this lecture..

Lecture 11: Queues

■Problem 8.4.1 (a) Let X_1, X_2, X_3 be the service time of the two customers and you. Denote the teller you chose by $Z = 1, 2$. Denote the exponential density function by $f_\lambda(x) = \lambda e^{-\lambda x}$. Then the probability we want is

$$\begin{aligned} p &= \frac{1}{2}P(X_1 + X_3 > X_2 | Z = 1) + \frac{1}{2}P(X_1 < X_2 + X_3 | Z = 2) \\ &= \frac{1}{2}\int_{x_1+x_3>x_2} f_\lambda(x_1)f_\mu(x_2)f_\lambda(x_3)dx_1dx_2dx_3 + \frac{1}{2}\int_{x_1<x_2+x_3} f_\lambda(x_1)f_\mu(x_2)f_\mu(x_3)dx_1dx_2dx_3 \\ &= \frac{1}{2}\left(\frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\mu}{\lambda+\mu}\right) + \frac{1}{2}\left(\frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\lambda}{\lambda+\mu}\right) \\ &= \frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{1}{2} \end{aligned}$$

(b) Suppose that $\lambda < \mu$. Then you always take $Z = 2$. Hence

$$\begin{aligned} p &= P(X_1 < X_2 + X_3 | Z = 2) \\ &= \frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\lambda}{\lambda+\mu} \end{aligned}$$

(c) If $X_1 < X_2$, then $Z = 1$. Otherwise, $Z = 2$. Therefore

$$\begin{aligned} p &= P(X_1 < X_2,\ X_1 + X_3 > X_2) + P(X_1 > X_2,\ X_1 < X_2 + X_3) \\ &= \frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\lambda\mu}{(\lambda+\mu)^2}. \end{aligned}$$

In particular, $p_{(\mathrm{a})} > p_{(\mathrm{b})} > p_{(\mathrm{c})}$.

■Problem 11.2.3 The waiting time W of a customer arriving at time 0 is the sum of service time of the $Q(0)$ people. Let $X_1, \ldots, X_n$ be the service time of the $Q(0) = n$ customers. Since $Q(0)$ has the stationary distribution, we have

$$P(W > x) = \sum_{n=1}^{\infty}(1-\rho)\rho^n P(X_1 + \cdots + X_n > x), \quad x \geq 0.$$

Since X_i has the exponential distribution with parameter μ, the distribution of $X_1 + \cdots + X_n$ is the gamma distribution. Therefore

$$P(X_1 + \cdots + X_n > x) = \int_x^\infty \frac{\mu^n t^{n-1}}{(n-1)!}e^{-\mu t}dt.$$

and

$$
\begin{aligned}
P(W > x) &= \int_x^\infty \sum_{n=1}^\infty (1-\rho)\rho^n \frac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!} dt \\
&= \int_x^\infty (1-\rho)\lambda \left(\sum_{n=1}^\infty \frac{(\lambda t)^{n-1}}{(n-1)!} \right) e^{-\mu t} dt, \quad \lambda = \rho\mu, \\
&= (1-\rho)\lambda \int_x^\infty e^{-(\mu-\lambda)t} dt \\
&= (1-\rho)\lambda \frac{e^{-(\mu-\lambda)x}}{\mu-\lambda} \\
&= \rho e^{-(\mu-\lambda)x}.
\end{aligned}
$$

We have $P(W \le x) = 1 - \rho e^{-(\mu-\lambda)x}$. In particular, $P(W = 0) = P(W \le 0) = 1 - \rho$.

■Problem 11.3.1 The problem is to determine the mean of $Q(D)$. By Theorem 13.3.5 of PRP, the generating function of $Q(D)$ is given by

$$
G(s) = (1-\rho)(s-1)\frac{M_S(\lambda(s-1))}{s - M_S(\lambda(s-1))},
$$

where $M_S(\theta)$ is the moment generating function of a typical service time. In this problem, the service time is the constant d. Therefore $M_S(\theta) = e^{\theta d}$, and

$$
G(s) = (1-\rho)(s-1)\frac{e^{\lambda(s-1)d}}{s - e^{\lambda(s-1)d}} = (1-\rho)(s-1)\frac{e^{\rho(s-1)}}{s - e^{\rho(s-1)}},
$$

where $\rho = \lambda d$ is used. The mean length of $Q(D)$ is $G'(1)$. By Taylor's expansion around $s = 1$, we obtain

$$
\begin{aligned}
G(1+r) &= (1-\rho)r \frac{1 + \rho r + O(r^2)}{1 + r - 1 - \rho r - \frac{1}{2}\rho^2 r^2 + O(r^3)} \\
&= (1-\rho)\frac{1 + \rho r + O(r^2)}{1 - \rho - \frac{1}{2}\rho^2 r + O(r^2)} \\
&= 1 + \left(\rho + \frac{1}{2}\frac{\rho^2}{1-\rho} \right) r + O(r^2) \\
&= 1 + \frac{1}{2}\frac{\rho(2-\rho)}{1-\rho} r + O(r^2).
\end{aligned}
$$

Hence $G'(1) = \frac{1}{2}\rho(2-\rho)/(1-\rho)$.

Lecture 12

■Problem 2 Typo: the definition of Z should be $Z = \sum_{i=1}^{N} b_{i-1}(W_{t_i} - W_{t_{i-1}})$.

Denote $Z = Z_N$ and $\mathcal{F}_t = \{W_s\}_{s \le t}$. By induction,

$$\begin{aligned} E[Z_N] &= E[Z_{N-1} + b_{N-1}(W_{t_N} - W_{t_{N-1}})] \\ &= E[Z_{n-1}] + E[b_{N-1}E[W_{t_N} - W_{t_{N-1}}|\mathcal{F}_{t_{N-1}}]] \\ &= E[Z_{n-1}] \\ &= E[Z_0] = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} V[Z_N] &= E[(Z_{N-1} + b_{N-1}(W_{t_N} - W_{t_{N-1}}))^2] \\ &= E[E[(Z_{N-1} + b_{N-1}(W_{t_N} - W_{t_{N-1}}))^2|\mathcal{F}_{t_{N-1}}]] \\ &= E[Z_{N-1}^2 + b_{N-1}^2(t_N - t_{N-1})] \\ &= \sum_{i=1}^{N} E[b_{i-1}^2](t_i - t_{i-1}). \end{aligned}$$

■Problem 4 The OU process is represented as

$$X_t = e^{-\theta t}\left(X_0 + \sigma \int_0^t e^{\theta s} dW_s\right).$$

Thus, given $X_0 = x$, the mean is $E[X_t] = e^{-\theta t}x \to 0$ as $t \to \infty$, and the variance is

$$\begin{aligned} V[X_t] &= E[(X_t - e^{-\theta t}x)^2] \\ &= \sigma^2 e^{-2\theta t} E\left[\left(\int_0^t e^{\theta s} dW_s\right)^2\right] \\ &= \sigma^2 e^{-2\theta t} \int_0^t e^{2\theta s} ds \quad \text{(Itô isometry)} \\ &= \sigma^2 \frac{1 - e^{-2\theta t}}{2\theta} \\ &\to \frac{\sigma^2}{2\theta} \quad (t \to \infty). \end{aligned}$$

Thus $N(0, \sigma^2/(2\theta))$ must be the stationary distribution. Indeed, if $X_0 \sim N(0, \sigma^2/(2\theta))$, then $X_t = e^{-\theta t}(X_0 + \sigma \int_0^t e^{\theta s} dW_s)$ is the sum of independent random variables having $N(0, e^{-2\theta t}\sigma^2/(2\theta))$ and $N(0, \sigma^2(1 - e^{-2\theta t})/(2\theta))$. Therefore $X_t \sim N(0, \sigma^2/(2\theta))$.

■Problem 5 (a) $d(W_t^3) = 3W_t^2 dW_t + 3W_t dt$. Thus $W_t^3 = 3\int_0^t W_s^2 dW_s + 3\int_0^t W_s ds$.
(b) $dF(W_t) = f(W_t)dW_t + (1/2)f'(W_t)dt$. Thus $F(W_t) = \int_0^t f(W_s)dW_s + (1/2)\int_0^t f'(W_s)ds$.

■Problem 7 For any $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\begin{aligned} E[df(X_t)] &= E\left[\sum_i (\partial_i f) dX_i(t) + (1/2)\sum_{i,j}(\partial_i\partial_j f)(dX_i(t))(dX_j(t))\right] \\ &= E\left[\sum_i (\partial_i f)\mu_i + (1/2)\sum_{i,j}(\partial_i\partial_j f)\sum_a \sigma_{ia}\sigma_{ja}\right] dt \end{aligned}$$

By using $E[f(X_t)] = \int f(y)p(t,y)dy$ and the integral-by-parts formula, we have

$$\int f(y)\partial_t p dy = \int f(y)\left[-\sum_i (\mu_i p) + (1/2)\sum_{i,j}\partial_i\partial_j\{\sum_a \sigma_{ia}\sigma_{ja}p\}\right] dy.$$

Since f is arbitrary, the result follows.

■Problem 9 The drift term is

$$\mu = \frac{1}{2g}\left(\log\frac{\pi_*}{\sqrt{g}}\right)' + \frac{1}{2\sqrt{g}}\left(\frac{1}{\sqrt{g}}\right)' = \frac{1}{2g}\left(\frac{\pi'}{\pi} - \frac{g'}{2g}\right) - \frac{g'}{4g^2} = \frac{\pi'}{2g\pi} - \frac{g'}{2g^2}$$

The diffusion term is $\sigma = 1/\sqrt{g}$. Then the right hand side of the forward equation is

$$\begin{aligned} -(\mu\pi)' + \frac{1}{2}(\sigma^2\pi)'' &= -\left(\frac{\pi'}{2g} - \frac{g'\pi}{2g^2}\right)' + \frac{1}{2}\left(\frac{\pi}{g}\right)'' \\ &= -\left(\frac{\pi'}{2g} - \frac{g'\pi}{2g^2}\right)' + \frac{1}{2}\left(\frac{\pi'}{g} - \frac{g'\pi}{g^2}\right)' \\ &= 0. \end{aligned}$$