Theory of Stochastic Processes 2017 S1S2, Final Exam

July 20

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- This examination paper consists of 6 questions.
- This is an open-book and open-note examination.
- Write your answers in English in the answer sheets provided. Exception: write your name in Japanese characters (if you have).
- The time allowed is 90 minutes.

Q 1. Complete the following sentences by selecting appropriate words from the list below.

(10 marks)

Sentences

A discrete-time stochastic process $\{X_n\}_{n\geq 0}$ taking values in a finite or countable set S is called a (1) if the conditional probability distribution of X_{n+1} given $X_0,\ldots,X_n$ depends only on X_n. It is further said to be reversible if it satisfies the (2) equation $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$, where p_{ij} denotes the (3) and π_i denotes the (4).

List of words

(A) Itô process, (B) death process, (C) state space, (D) martingale, (E) stationary distribution, (F) normal distribution, (G) queue, (H) win-win relationship, (I) detailed balance, (J) new balance, (K) transition probability, (L) Markov chain, (M) supply chain.

Q 2. Define a (discrete-time) Markov chain on the state space $\{1,2,3\}$ by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

(a) Find a stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$. (10 marks)

(b) Find the mean recurrence time μ_i for each state i. (5 marks)

Q 3. Consider a M(λ)/M(μ)/1 queue with $\rho = \lambda/\mu < 1$, and suppose that the number $Q(0)$ of people in the queue at time 0 has the stationary distribution.

(a) Find the mean value of $Q(0)$. (10 marks)

(b) Find the mean queue length at moments of departure. (10 marks)

Q 4. Let Z_n be a white noise with the spectral density $1/(2\pi)$. Define X_n by

$$X_n = (\cos\alpha)Z_n + (\sin\alpha)Z_{n-1},$$

where $\alpha \in [0, \pi/2]$ is a constant.

(a) Find the spectral density function $f(\lambda)$ of X_n. (10 marks)

(b) Find α maximizing $f(0)$. (5 marks)

Q 5. Let $S = \{S_n\}_{n\geq 0}$ be a symmetric simple random walk.

(a) Show that $X_n = nS_n - \sum_{m=1}^{n-1} S_m$ is a martingale (with respect to S). (10 marks)

(b) Show that $Y_n = |S_n|$ is a submartingale (with respect to S). (10 marks)

Q 6. Let $W = \{W_t\}_{t\geq 0}$ be the standard Brownian motion. Let $X_t = e^{\theta W_t - \theta^2 t/2}$, where θ is a real number.

(a) Use Itô's formula to show that X_t is a martingale (with respect to W). (10 marks)

(b) Show that X_t is a diffusion process and write down its forward equation. (10 marks)