Theory of Stochastic Processes
12. Diffusion processes

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Handouts & Announcements

Handouts:

- Slides (this one)
- Lecture notes for today

Announcements:

- Next week I will solve a part of recommended problems. If you have specific requests, please let me know by Tuesday.
- There is a course questionnaire. Please answer it today.
- About the final exam (reminder)
  - The final exam is held on July 20 (Thu), 10:25–, 90 min.
  - Contact me if you cannot attend it due to some unavoidable reasons.
  - The exam is open-book and open-note. Write your answer in English.
  - The exam will cover the whole material up to today (July 6). At least one question will be from the topics before the midterm exam.

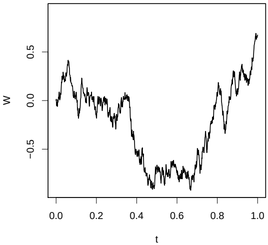
Outline today

1. Diffusion processes
   - Brownian motion
   - Itô calculus
   - Diffusion processes
2. Recommended problems

$\longrightarrow$ Let us look at highlights. Details will be followed.

Brownian motion

A Brownian motion $\{W_t\}_{t \geq 0}$ is a continuous-time version of the symmetric random walk.



Properties:

- a Markov process
- a martingale
- derive a heat equation.

Stochastic integral

Continuous-time betting game:

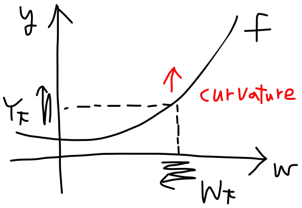
- Your initial capital is $X_0 > 0$.
- The stock price W_t is assumed to be a Brownian motion.
- If you hold b_t stocks during the time interval $[t, t + \Delta t]$, you will obtain $b_t(W_{t+\Delta t} - W_t)$ as a profit.
- Then your capital at time t will be

$$X_t - X_0 = \int_0^t b_s dW_s.$$

The right hand side is called the stochastic integral.

Itô's formula

- Remember that if $y_t = f(w_t)$ and w_t is a smooth function of t, then
$$\frac{dy_t}{dt} = f'(w_t)\frac{dw_t}{dt},$$
or simply $dy_t = f'(w_t)dw_t$.
- Itô's formula: if $Y_t = f(W_t)$ and W_t is the Brownian motion, then
$$dY_t = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$
The second term is caused by fluctuation of the Brownian motion.



Diffusion processes and their application

A diffusion process is a solution to the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

Properties:

- a Markov process
- derives a forward equation (Fokker-Planck equation)

Applications:

- modelling of financial data
- stochastic control
- Langevin Monte Carlo $\rightarrow$ today

Recommended problems

Recommended problems:

- Exercises in the lecture notes.