Theory of Stochastic Processes
13. Review

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Handouts & Announcements

Handouts:

- Slides (this one)
- Solved problems

Announcements:

- About the final exam (reminder)
  - The final exam is held on July 20 (Thu), 10:25–, 90 min.
  - Contact me if you cannot attend it due to some unavoidable reasons.
  - The exam is open-book and open-note. Write your answer in English.
  - The exam will cover the whole material up to July 6. At least one question will be from the topics before the midterm exam.

Outline today

1. Review of last week's material
2. Solved problems

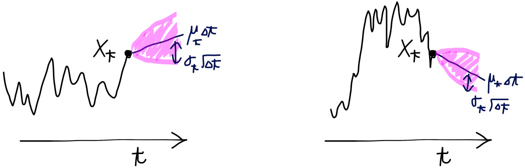
Review: Itô calculus

- Let W_t be the standard Brownian motion.
- An Itô process X_t is defined by

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

- The conditional distribution given $\mathcal{F}_t = \{W_s\}_{s \le t}$ is approximately

$$X_{t+\Delta t} | \mathcal{F}_t \sim N(\mu_t \Delta t, \sigma_t^2 \Delta t).$$



- X_t is a martingale if $\mu_t = 0$.
- Itô's formula:

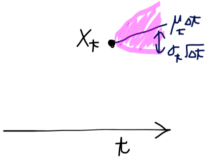
$$d\{f(t, X_t)\} = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2.$$

Diffusion processes

- A diffusion process is a solution to the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

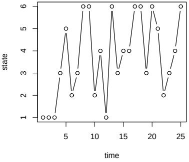
- X_t is Markov.

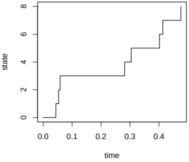


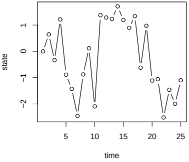
The future behavior depends only on the present value.

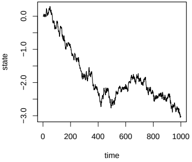
Markov processes

We encountered many examples of Markov processes.

<table>
<tr><th></th><th>discrete time</th><th>continuous time</th></tr>
<tr><td>discrete state</td><td>

Markov chain</td><td>

Poisson process</td></tr>
<tr><td>continuous state</td><td>

AR(1) process</td><td>

diffusion process</td></tr>
</table>

The forward equation

- Denote the transition density from $X_s = x$ to $X_t = y$ by $p(t, y|s, x)$.
- The forward equation (Fokker-Planck equation) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(\mu p) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2 p).$$

Key exercise

If μ and σ are constant, then $X_t \sim N(\mu t, \sigma^2 t)$. Write down the transition density $p(t, y) = p(t, y|0, 0)$ and check that the forward equation holds.

- Proof sketch: Itô's formula implies

$$dE[f(X_t)] = E[f'(X_t)\mu + (1/2)f''(X_t)\sigma^2]dt$$

for any function f. Since $E[f(X_t)] = \int f(y)p(t, y|s, x)dy$, we have

$$\int f(y)\frac{\partial p}{\partial t}dy = \int \{f'(y)\mu + (1/2)f''(y)\}pdy.$$

The result follows from the integral-by-parts formula.

Langevin Monte Carlo

- The diffusion process

$$dX_t = \frac{1}{2}(\log \pi_*(X_t))' dX_t + dW_t$$

  has the stationary distribution $\pi(x) = \pi_*(x)/Z$.
- Indeed, $p(t, y) = \pi(y)$ satisfies the forward equation:

$$-\left(\frac{1}{2}(\log \pi_*)'\pi\right)' + \frac{1}{2}\pi'' = -\left(\frac{1}{2}\pi'\right)' + \frac{1}{2}\pi'' = 0.$$

- MCMC using the diffusion process is called Langevin Monte Carlo.

Metropolis-Adjusted Langevin algorithm (MALA)

- In practice, discretize the equation as

$$X_{t+\Delta t} = X_t + \mu(X_t)\Delta t + \Delta W_t, \quad \Delta W_t \sim N(0, \sqrt{\Delta t})$$

  (Euler-Maruyama scheme), where $\mu(x) = (1/2)(\log \pi_*)'(x)$.

- The conditional density of $X_{t+\Delta t}$ given $X_t = x$ is

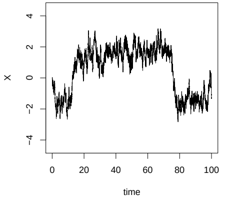
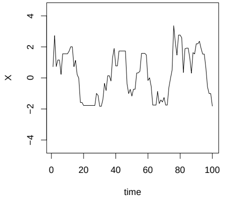
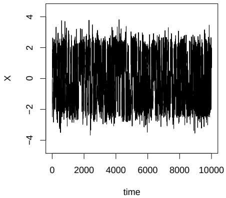
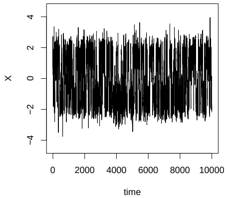
$$q(y|x) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(y - x - \mu(x)\Delta t)^2}{2\Delta t}\right).$$

  It is used as a proposal density for the Metropolis-Hastings algorithm.

- Specifically, the algorithm is
  1. Set an initial value x.
  2. Generate a random number y according to $q(y|x)$.
  3. Compute the acceptance probability $a = \min(1, \frac{\pi_*(y)q(x|y)}{\pi_*(x)q(y|x)})$.
  4. Genrate $U \sim U(0,1)$. If $U \leq a$, then $x \leftarrow y$. Otherwise, $x \leftarrow x$.
  5. Go to step 2.

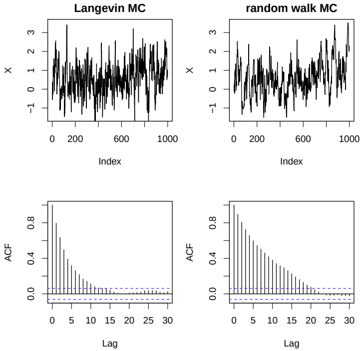
Example 1

- Let $\pi_*(x) = \dfrac{1}{\{1+(x-2)^2\}^3\{1+(x+2)^2\}^3}$.
- Sample paths

	$\Delta t = 0.01$ without MH	$\Delta t = 1$ with MH
$t \in [0, 100]$		
$t \in [0, 10000]$		

Example 2

- Let $\pi_*(x) = \phi(x)\Phi(x)$, $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$.
- Comparison of Langevin MC (left) and random walk MC (right), where the random walk MC uses $X_{t+\Delta t} = X_t + \Delta W_t$ as the proposal.



The underlined problems are solved. See the handout.

June 8 (Lec 8) Markov chain Monte Carlo

- §6.5: Problem <u>1</u>, <u>8*</u>, 9.
- §6.14: Problem 1.
- §6.15: Problem 2.

June 15 (Lec 9) Stationary processes

- Problem 1, <u>2</u>, 3, <u>4</u>, <u>5</u>, <u>6</u>.

June 22 (Lec 10) Martingales

- §12.1: Problem 1, 2, <u>3</u>, 4, <u>5</u>, <u>6</u>, 7*, 8, 9*.
- §12.2: Problem <u>1</u>, 2.

June 29 (Lec 11) Queues

- §8.4: Problem <u>1</u>, 2, 3, 4*, 5*.
- §11.2: Problem <u>3</u>, 7*.
- §11.3: Problem <u>1*</u> .

July 6 (Lec 12) Diffusion processes

- Problem 1, <u>2</u>, 3, <u>4</u>, <u>5</u>, 6, <u>7</u>, 8, <u>9</u>.

Final remark

Before

After

We have lost fresh flowers but obtained many leaves of knowledge!

Thanks