Theory of Stochastic Processes
9. Stationary processes

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Handouts & Announcements

Handouts:

- Slides (this one)
- Lecture notes for today (main material)
- Copy of Section 9.4

Announcements:

- If you have claims about the midterm exam, please contact me by today.

Outline today

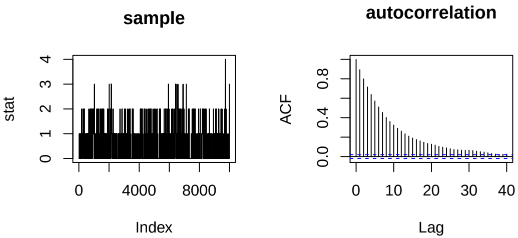
1. Review of last week's material
2. Stationary processes
   - Autocovariance
   - Spectral distribution
   - Causal processes
3. Recommended problems

Review of last week

Example (a toy example)

- Θ is the set of all length-20 strings consisting of 'A', 'G', 'C', 'T'.
- Let $\pi(\theta) \propto 2^{c(\theta)}$, where $c(\theta) = \sharp\{\text{“CAT” in } \theta\}$.
- Estimate $\sum c(\theta)\pi(\theta)$ by the Gibbs sampler.

A result: 0.534 (standard error 0.028)



An R code $\rightarrow$ web page

Remark: effective sample size

- The purpose of MCMC was to estimate $\mu = \sum g(\theta)\pi(\theta)$ by

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} g(\theta^{(n)}),$$

  where $\{\theta^{(n)}\}_{n=1}^{N}$ is an outcome of the constructed Markov chain.
- But how much is the error of $\hat{\mu}$?
- The standard error of an estimator $\hat{\mu}$ is given by

$$\mathrm{se}(\hat{\mu}) = \sqrt{\frac{\hat{V}}{N_{\mathrm{eff}}}}, \quad N_{\mathrm{eff}} = \frac{N}{\sum_{n=-\infty}^{\infty} \hat{\rho}(n)},$$

  where $\hat{V}$ is the sample variance of $\{g(\theta^{(n)})\}$ and $\hat{\rho}$ is an estimated autocorrelation function ($\rightarrow$ today's topic).
- The quantity N_{eff} is called the effective sample size.
- In the R language, the package 'coda' offers a function effectiveSize.

Metropolis-Hastings algorithm

- Given $X_n = i$, choose j according to a proposal matrix h_{ij}.
- Let $X_{n+1} = j$ with the acceptance probability a_{ij}, where

$$a_{ij} = \min\left(1, \frac{\pi_j h_{ji}}{\pi_i h_{ij}}\right),$$

and $X_{n+1} = i$ with probability $1 - a_{ij}$.

Exercise

Let $\Theta = \mathbb{Z} = \{0, \pm 1, \cdots\}$ and $\pi_i \propto e^{-i^2/2}$. Define a proposal matrix by

$$h_{ij} = \begin{cases} 1/3 & \text{if } j \in \{i-1, i, i+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the acceptance probability a_{ij} for each pair (i, j) with $h_{ij} \neq 0$.

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$\longrightarrow$ see the lecture notes. I will use blackboard.

Recommended problems

Recommended problems:

- Exercises in the lecture notes.