Oscillator decomposition of time series data

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Abstract

- We develop a method for decomposing time series data into oscillators (without band-pass filtering).

Background: neural oscillation
Neural oscillation

- neural time series: recording of brain activity
  - EEG, MEG, fMRI, fNIRS
- They are composed of several oscillation components.
  - delta: 0.5-4 Hz, alpha: 8-13 Hz, beta: 13-20 Hz

- Local Field Potential (LFP) from rat hippocampus

- Theta component (6-10 Hz) is clearly seen.
Phase of neural oscillation

- Phase of neural time series plays an important role in neural information processing.

- Phase reset occurs in response to external stimuli. (Makeig et al., 2012; Lopour et al., 2013)

- Theta phase precession in hippocampal LFP encodes the place. (O’Keefe and Recce, 1993)

- Phase synchronization is important for coordination of distant neural assemblies. (Buzsaki, 2011)
Conventional analysis with band-pass filtering

raw EEG

band-pass filtering

alpha (8-13 Hz)

beta (13-20Hz)

Hilbert transform

alpha phase, beta power, ...
Hilbert transform

\[ y(t) \mapsto y_H(t) = \int_{-\infty}^{\infty} \frac{y(\tau)}{t - \tau} d\tau \]

- analytical signal (complex time series)

\[ x(t) = y(t) + i y_H(t) \]

- In frequency domain,

\[ X(\omega) = \begin{cases} 2Y(\omega) & (\omega > 0) \\ Y(\omega) & (\omega = 0) \\ 0 & (\omega < 0) \end{cases} \]
Hilbert transform

- The phase $\phi(t)$ and amplitude $r(t)$ are defined by the angle and absolute value of $x(t)$.

$$\phi(t) = \text{arg } x(t), \quad r(t) = |x(t)|$$

- When $y(t) = \cos \omega t$,

$$y_H(t) = \sin \omega t, \quad x(t) = \exp(i \omega t)$$

$$\phi(t) = \omega t, \quad r(t) = 1$$
Conventional analysis with band-pass filtering

raw EEG

band-pass filtering

alpha (8-13 Hz)

beta (13-20Hz)

Hilbert transform

alpha phase, beta power, ...
Limitations of band-pass filtering

- requires subjective selection of filters
  - definition of frequency band varies among studies..
  - alpha = 8-13 Hz ? 9-12 Hz ?
  - alpha/beta frequency may depend on individuals
- does not account for measurement noise
- distorts the waveform
  - Siapas et al. (2005), methods section
    - since the decomposition is exact even when the input contains a mixture of signal and noise, both enter the instantaneous phase and amplitude components, and thus any denoising must occur at the earlier filtering step.
    - This dictates the use of narrow band filters in conjunction with this method that can distort the input signal waveform.

→ We overcome them by state space model approach
Oscillator decomposition
State space model

- general framework for estimating hidden dynamics from time series data

\[
x_{t+1} = f(x_t, v_t), \quad v_t \sim p(v_t)
\]
\[
y_t = h(x_t, w_t), \quad w_t \sim p(w_t)
\]

- \(x_t\): state (unobserved), \(y_t\): data (observed)

- The posterior \(p(x_s \mid y_1, \cdots, y_t)\) can be computed sequentially
  - filtering \((s = t)\), smoothing \((s < t)\), prediction \((s > t)\)
Gaussian linear state space model

- Special class of state space models

\[ x_{t+1} = Fx_t + Gv_t, \quad v_t \sim N(0, Q) \]
\[ y_t = Hx_t + w_t, \quad w_t \sim N(0, R) \]

- The posteriors \( p(x_s \mid y_1, \cdots, y_t) \) are Gaussian and they are obtained by matrix computation (Kalman filter, Kalman smoother).
cf. Bayesian seasonal adjustment

- State space models can be used to decompose time series

- Kitagawa and Gersch (1984) developed a method for decomposing economic time series into trend and seasonal components based on a Gaussian linear state space model.

\[
T_{t+1} = F_T T_t + G_T v_t, \quad S_{t+1} = F_S S_t + G_S v_t, \quad v_t \sim \mathcal{N}(0, Q)
\]

\[
y_t = T_t + S_t + w_t, \quad w_t \sim \mathcal{N}(0, R)
\]

decomposition is obtained by smoothing

\[
y_t = T_{t|N} + S_{t|N} + w_{t|N}
\]

\[
T_{t|N} = \mathbb{E}[T_t \mid Y_1, \cdots, Y_N], \quad S_{t|N} = \mathbb{E}[S_t \mid Y_1, \cdots, Y_N]
\]
cf. Bayesian seasonal adjustment

• Construction Housing Starts North data
  ▶ Kitagawa and Gersch (1984)

• Seasonal adjustment improves prediction accuracy.
Model building

- assumption: several oscillators underlie the time series data
- Each oscillator rotates on a 2-dim. plane with fluctuation.

The observed time series is regarded as the sum of the projections of the oscillators (plus noise).
- trigonometric function = projection of circular motion
- cf. Hilbert transform
System model for oscillators

- We describe each oscillator by a Gaussian linear model
  - $\Delta t$: sampling period

$$
\begin{pmatrix}
  x_{t+1,1} \\
  x_{t+1,2}
\end{pmatrix}
\sim
\mathcal{N}
\left(
  a
  \begin{pmatrix}
    \cos(2\pi f \Delta t) & -\sin(2\pi f \Delta t) \\
    \sin(2\pi f \Delta t) & \cos(2\pi f \Delta t)
  \end{pmatrix}
  \begin{pmatrix}
    x_{t,1} \\
    x_{t,2}
  \end{pmatrix},
  \begin{pmatrix}
    \sigma^2 & 0 \\
    0 & \sigma^2
  \end{pmatrix}
\right)
$$

- $f$: frequency, $a$: regularity ($0 < a < 1$), $\sigma^2$: power
- In each time step, the oscillator rotates through an angle $2\pi f \Delta t$ with fluctuation.
- At each time point, the phase is defined as

$$
\phi_t = \tan^{-1} \frac{x_{t,2}}{x_{t,1}}
$$
State space model for oscillator decomposition

- $K$: number of oscillators

\[
\begin{pmatrix}
  x_{t+1,1}^{(k)} \\
  x_{t+1,2}^{(k)}
\end{pmatrix}
\sim \mathcal{N}
\left(\begin{pmatrix}
  a_k \begin{pmatrix}
    \cos(2\pi f_k \Delta t) & - \sin(2\pi f_k \Delta t) \\
    \sin(2\pi f_k \Delta t) & \cos(2\pi f_k \Delta t)
  \end{pmatrix} \begin{pmatrix}
    x_{t,1}^{(k)} \\
    x_{t,2}^{(k)}
  \end{pmatrix},
  \begin{pmatrix}
    \sigma_k^2 & 0 \\
    0 & \sigma_k^2
  \end{pmatrix}
\end{pmatrix}
\]

$(k = 1, \ldots, K)$

\[
y_t \sim \mathcal{N}\left(\sum_{k=1}^{K} x_{t,1}^{(k)}, \tau^2\right)
\]

- Based on the observation $y_1, \ldots, y_T$, the posterior of $x_t$ is calculated by the Kalman smoother.

$\rightarrow$ decomposition of $y_t$ into $x_{t,1}^{(1)}, \ldots, x_{t,1}^{(K)}$ is obtained

- The phase of each oscillator is also estimated (with credible interval).
parameter estimation & model selection

- Model parameters $\theta_K = (f_1, \cdots, f_K, a_1, \cdots, a_K, \sigma_1^2, \cdots, \sigma_K^2, \tau^2)$ and number of oscillators $K$ are determined in a data-driven manner.

- Model parameters are estimated by empirical Bayes method.
  - maximization of marginal likelihood

$$\hat{\theta}_K = \arg \max_{\theta_K} \log p(y \mid \theta_K)$$

- Number of oscillators are determined by minimizing Akaike Bayes Information Criterion (ABIC).

$$\text{ABIC}_K = -2 \log p(y \mid \hat{\theta}_K) + 2(3K + 1)$$

→ natural decomposition of time series data is attained
Remark: initial value setting by AR model

- We select the initial value for parameter estimation by using the AR model

- AR model can be viewed as a sum of oscillation components
  - each characteristic root $z$ represents an oscillator with frequency $\arg z$

- examine the location and sharpness of each peak in the fitted AR spectrum
  $\rightarrow a_k, f_k$

- fit spectrum to periodogram based on the Whittle likelihood
  $\rightarrow \sigma_k^2$
Simulation results
Simulation setting

- We generated time series with three oscillators from the following model.
  - data length: 1000
  - sampling frequency: 200 Hz
  - 25 Hz oscillator & 50 Hz oscillator & 75 Hz oscillator

\[
\begin{align*}
\log r^{(k)}_{t+1} &\sim N(b_k \log r^{(k)}_t, 0.1^2) \quad (k = 1, \cdots, K) \\
\phi^{(k)}_{t+1} &\sim \text{vM}(\phi^{(k)}_t + \Delta \phi_k, \kappa_k) \quad (k = 1, \cdots, K) \\
y_t &\sim N\left(\sum_{k=1}^{K} r^{(k)}_t \cos \phi^{(k)}_t, \tau^2\right)
\end{align*}
\]

- \(\text{vM}(\mu, \kappa)\) is the von Mises distribution with center \(\mu\) and concentration \(\kappa\).
ABIC takes minimum at $K = 3$.

→ The proposed method selects the correct number of oscillators

parameter estimates for $K = 3$

\[
\hat{f}_1 = 25.22, \hat{f}_2 = 50.40, \hat{f}_3 = 75.10
\]

→ The proposed method estimates the frequencies accurately
decomposition

- original time series
decomposition

- obtained three oscillation components
  - with 67% credible intervals
mean squared errors of phases $\phi_t^{(1)}, \phi_t^{(2)}, \phi_t^{(3)}$ (in radian)

<table>
<thead>
<tr>
<th></th>
<th>$\phi^{(1)}$</th>
<th>$\phi^{(2)}$</th>
<th>$\phi^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>proposed</td>
<td>0.23</td>
<td>0.16</td>
<td>0.13</td>
</tr>
<tr>
<td>Hilbert</td>
<td>0.27</td>
<td>0.37</td>
<td>0.48</td>
</tr>
</tbody>
</table>

The proposed method has better accuracy in phase estimation.

- Hilbert transform uses band-pass filters with passband 20-30 Hz, 40-60 Hz, and 60-90 Hz.
phase reset detection

- time series with two oscillators (10 Hz & 30 Hz)
  - several phase resets in 10 Hz oscillator
- It was decomposed into 9.78Hz and 29.68Hz oscillators.
- upper: estimated phase, lower: true phase

The proposed method detects phase resets successfully.
ripple detection

- time series with ripples (intermittent oscillations)
  \[ g(t) = A \cos^8(2\pi f_0 t) \sin(2\pi f_1 t) + 10 \cos(2\pi f_2 t) \]

- decomposition result

The proposed method detects ripples successfully.
Application to real data
Application to hippocampal LFP

- Local Field Potential from rat hippocampus
  - Mizuseki et al. (2009)
- sampling frequency: 125 Hz
- data length: 250 (two seconds)
Decomposition (hippocampal LFP)

- decomposed into six oscillators

\[ \hat{f}_1 = 6.45, \hat{f}_2 = 8.00, \hat{f}_3 = 15.79, \hat{f}_4 = 34.63, \hat{f}_5 = 40.35, \hat{f}_6 = 55.44 \]

- Their phases develop regularly.
Interpretation (hippocampal LFP)

- **Two** oscillators (6.45 Hz, 8.00 Hz) correspond to the theta rhythm.
  - Band-pass filtering with theta band cannot separate these oscillators.
- Another oscillator (15.79 Hz) corresponding to the alpha rhythm likely exists.
- Higher frequency oscillators are also evident.
  - Future research may investigate the role of these oscillators, including the potential for their phases to encode internal information.

In this way, neural oscillators underlying the hippocampal LFP are extracted in a data-driven manner.
Application to Canadian Lynx data

- annual number of the Canadian lynxes (1821 - 1934)

Among six oscillators, one oscillator (5 years period) showed phase reset around 1850.

In this way, the phase dynamics may provide an interesting insight from time series data.
Extension to multivariate time series
Extension to multivariate time series

- Several oscillators underlie multivariate time series.
  - neural oscillators & electrodes
  - earthquake origins & stations
- model assumption: projection of each oscillator superposes on each variable.
  - projection describes amplitude & phase modulation
State space model

\[
\begin{pmatrix}
    x_{t+1,1}^{(k)} \\
    x_{t+1,2}^{(k)}
\end{pmatrix} = a_k \begin{pmatrix}
    \cos(2\pi f_k \Delta t) & - \sin(2\pi f_k \Delta t) \\
    \sin(2\pi f_k \Delta t) & \cos(2\pi f_k \Delta t)
\end{pmatrix} \begin{pmatrix}
    x_{t,1}^{(k)} \\
    x_{t,2}^{(k)}
\end{pmatrix} + \begin{pmatrix}
    v_{t,1}^{(k)} \\
    v_{t,2}^{(k)}
\end{pmatrix},
\]

\[
y_{t,j} = \sum_{k=1}^{K} (c_{jk,1} x_{t,1}^{(k)} + c_{jk,2} x_{t,2}^{(k)}) + w_{t,j},
\]

\[
(v_{t,1}^{(k)}, v_{t,2}^{(k)})^\top \sim N_2(0, \sigma_k^2 I),
\]

\[
(w_{t,1}, \cdots, w_{t,J})^\top \sim N_J(0, \tau^2 I).
\]
System model

- same with the univariate case

\[
\begin{pmatrix}
    x_{t+1,1}^{(k)} \\
    x_{t+1,2}^{(k)}
\end{pmatrix} = a_k \begin{pmatrix}
    \cos(2\pi f_k \Delta t) & -\sin(2\pi f_k \Delta t) \\
    \sin(2\pi f_k \Delta t) & \cos(2\pi f_k \Delta t)
\end{pmatrix} \begin{pmatrix}
    x_{t,1}^{(k)} \\
    x_{t,2}^{(k)}
\end{pmatrix} + \begin{pmatrix}
    v_{t,1}^{(k)} \\
    v_{t,2}^{(k)}
\end{pmatrix},
\]

\[
\begin{pmatrix}
    v_{t,1}^{(k)} \\
    v_{t,2}^{(k)}
\end{pmatrix} \sim N_2(0, \sigma_k^2 I),
\]
Observation model

\[ y_{t,j} = \sum_{k=1}^{K} (c_{jk,1} x_{t,1}^{(k)} + c_{jk,2} x_{t,2}^{(k)}) + w_{t,j} \]

\[ (w_{t,1}, \cdots, w_{t,J})^\top \sim N_J(0, \tau^2 I) \]

- \( y_{t,j} \) = sum of inner products of \( x_t^{(k)} \) and \( c_{jk} = (c_{jk,1}, c_{jk,2}) \)
  - \( c_{1k} = (1, 0) \) for identifiability
- length and angle of \( c_{jk} \) describe amplitude and phase modulation
  - e.g., shorter length for farther sensors
Application to north/south sunspot data

- The number of sunspots on northern/southern hemisphere
  - from ICSU World Data System

South sunspot numbers seem to have a slight delay compared to north sunspot numbers.
Phase delay in north/south sunspot data

- Our method decomposes this data into six oscillators.
- The oscillator with 11.76 years period is dominant in power.
  - It corresponds well to the well-known period in sunspot numbers.
- Projection vector: $(1, 0)$ for north and $(0.82, 0.32)$ for south → south follows north with a delay of 0.69 years!!
We developed a method for decomposing time series data into oscillators (without band-pass filtering).
References
